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Bandura A.I.

PROPERTIES OF POSITIVE CONTINUOUS FUNCTIONS IN $\mathbb{C}^n$

The properties of classes $Q^n_{\mathbf{b}}$ and Q of positive continuous functions are investigated. We prove that some compositions of functions from Q belong to class $Q^n_{\mathbf{b}}$. A relation between functions from these classes is established.

Key words and phrases: positive function, continuous function, several complex variables.

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Introduction

Introducing entire functions of bounded L-index in direction (see [1]) we have to impose additional conditions to a continuous function $L: \mathbb{C}^n \to \mathbb{R}_+$. We suppose that $L \in Q^n_{\mathbf{b}}$ (see below (5)). It is necessary to establish criteria of boundedness of L-index in direction and to apply L-index for solutions of partial differential equations or for entire functions with "plane" zeros [3].

Such conditions describe a behavior of slice function $L(z^0 + t\mathbf{b})$, $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$. It provides that function L does not rapidly change as $|z| \to \infty$. In one-dimensional case Sheremeta M.M. [5] used a class Q of positive continuous functions $l = l(t)$, $t \in \mathbb{C}$, satisfying some additional conditions. In fact, $l(t) = \ln|t|$, $l(t) = |t|^\alpha$, $\alpha \in \mathbb{R}_+$ belong to Q.

It is interesting: what are examples of functions from $Q^n_{\mathbf{b}}$? To answer the question we consider compositions of functions from Q. Thus, it is a natural question: how to build a function $L \in Q^n_{\mathbf{b}}$ by a function $l \in Q$?

1 Preliminaries and denotations

For $\eta > 0$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{0\}$ and a positive continuous function $L: \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta) = \inf\left\{ \frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})} \right\}, \tag{1}$$

$$\lambda_1^{\mathbf{b}}(z, \eta) = \inf\{\lambda_1^{\mathbf{b}}(z, t_0, \eta) : t_0 \in \mathbb{C}\}, \qquad \lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta) : z \in \mathbb{C}^n\}, \tag{2}$$

and

$$\lambda_2^{\mathbf{b}}(z, t_0, \eta) = \sup\left\{ \frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})} \right\}, \tag{3}$$

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$$\lambda_2^{\mathbf{b}}(z,\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta): t_0\in\mathbb{C}\},\qquad \lambda_2^{\mathbf{b}}(\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,\eta): z\in\mathbb{C}^n\}. \tag{4}$$

By $Q^n_{\mathbf{b}}$ we denote the class of functions L, which for all $\eta\geq 0$ satisfy the condition

$$0<\lambda_1^{\mathbf{b}}(\eta)\leq\lambda_2^{\mathbf{b}}(\eta)<+\infty. \tag{5}$$

For a positive continuous function $l(t)$ for $t\in\mathbb{C}$ and $t_0\in\mathbb{C}$, $\eta>0$ we denote $\lambda_1(t_0,\eta)\equiv\lambda_1^{\mathbf{b}}(0,t_0,\eta)$ and $\lambda_2(t_0,\eta)\equiv\lambda_2^{\mathbf{b}}(0,t_0,\eta)$ in the case $z=0$, $\mathbf{b}=1$, $n=1$, $L\equiv l$, and

$$\lambda_1(\eta)=\inf\{\lambda_1(t_0,\eta): t_0\in\mathbb{C}\},\qquad \lambda_2(\eta)=\sup\{\lambda_2(t_0,\eta): t_0\in\mathbb{C}\}.$$

As in [5], by Q we denote the class of positive continuous functions $l(t)$, $t\in\mathbb{C}$, which satisfy the condition: $0<\lambda_1(\eta)\leq\lambda_2(\eta)<+\infty$ for all $\eta\geq 0$. In particular, $Q=Q^1_1$.

2 Elementary properties of functions from $Q^n_{\mathbf{b}}$

Investigating the properties of entire functions of bounded L-index in direction we obtained following propositions about class $Q^n_{\mathbf{b}}$.

Lemma 1 ([1]). *If $L\in Q^n_{\mathbf{b}}$, then $L\in Q^n_{\theta\mathbf{b}}$ for every $\theta\in\mathbb{C}\setminus\{0\}$, and if $L\in Q^n_{\mathbf{b}_1}$ and $L\in Q^n_{\mathbf{b}_2}$ then $L\in Q^n_{\mathbf{b}_1+\mathbf{b}_2}$ for any $\mathbf{b}_1,\mathbf{b}_2\in\mathbb{C}^n$.*

For $l\in Q$ we denote

$$l_1(t,w)=(|t|+|w|+1)l(tw),\qquad l_2(t,w)=(|w|+1)l(tw),\qquad l_3(t,w)=(|t|+1)l(tw),$$

where $t,w\in\mathbb{C}$.

Lemma 2 ([2]). *If $l\in Q$, then $\forall\mathbf{b}\in\mathbb{C}^2$ $l_1\in Q^2_{\mathbf{b}}$, $l_2\in Q^2_{\mathbf{b}_1}$, $l_3\in Q^2_{\mathbf{b}_2}$, where $\mathbf{b}_1=(1,0)$, $\mathbf{b}_2=(0,1)$.*

For $l\in Q$ we denote $l_4(z)=l(|\langle z,m\rangle|)$, where $z\in\mathbb{C}^n$, $m\in\mathbb{C}^n$.

Lemma 3 ([4]). *If $l\in Q$, then $l_4\in Q^n_{\mathbf{b}}$ for every $m\in\mathbb{C}^n$ and every $\mathbf{b}\in\mathbb{C}^n$.*

For $l\in Q$ we denote $l_5(z)=l(|z|)$, $z\in\mathbb{C}^n$.

Lemma 4 ([4]). *If $l\in Q$, then $l_5\in Q^n_{\mathbf{b}}$ for every $\mathbf{b}\in\mathbb{C}^n$.*

It is easy to see that Lemmas 2, 3, 4 propose possible ways to construct a function $L\in Q^n_{\mathbf{b}}$ by a function $l\in Q$. Below we prove a generalization of Lemma 2 for $\mathbb{C}^n$ (see Theorem 1).

Let $L^*(z)$ be a positive continuous function in $\mathbb{C}^n$. The denotation $L\asymp L^*$ means that for some $\theta_1,\theta_2\in\mathbb{R}_+$, and for all $z\in\mathbb{C}^n$ the inequalities $\theta_1 L(z)\leq L^*(z)\leq\theta_2 L(z)$ hold.

Lemma 5. *If $L\in Q^n_{\mathbf{b}}$ and $L\asymp L^*$, then $L^*\in Q^n_{\mathbf{b}}$.*

Proof. Using the definition of $Q^n_{\mathbf{b}}$, we have

$$\begin{aligned}\inf_{z\in\mathbb{C}^n}\inf_{t_0\in\mathbb{C}}\inf&\left\{\frac{L^*(z+t\mathbf{b})}{L^*(z+t_0\mathbf{b})}: |t-t_0|\leq\frac{\eta}{L^*(z+t_0\mathbf{b})}\right\}\\ &\geq\inf_{z\in\mathbb{C}^n}\inf_{t_0\in\mathbb{C}}\inf\left\{\frac{\theta_1 L(z+t\mathbf{b})}{\theta_2 L(z+t_0\mathbf{b})}: |t-t_0|\leq\frac{\eta}{\theta_1 L(z+t_0\mathbf{b})}\right\}\\ &=\frac{\theta_1}{\theta_2}\inf_{z\in\mathbb{C}^n}\inf_{t_0\in\mathbb{C}}\inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}: |t-t_0|\leq\frac{\eta}{\theta_1 L(z+t_0\mathbf{b})}\right\}>0,\end{aligned}$$

because $L\in Q^n_{\mathbf{b}}$. Besides,

$$\begin{aligned}\sup_{z\in\mathbb{C}^n}\sup_{t_0\in\mathbb{C}}\sup&\left\{\frac{L^*(z+t\mathbf{b})}{L^*(z+t_0\mathbf{b})}: |t-t_0|\leq\frac{\eta}{L^*(z+t_0\mathbf{b})}\right\}\\ &\leq\sup_{z\in\mathbb{C}^n}\sup_{t_0\in\mathbb{C}}\sup\left\{\frac{\theta_2 L(z+t\mathbf{b})}{\theta_1 L(z+t_0\mathbf{b})}: |t-t_0|\leq\frac{\eta}{\theta_1 L(z+t_0\mathbf{b})}\right\}\\ &=\frac{\theta_2}{\theta_1}\sup_{z\in\mathbb{C}^n}\sup_{t_0\in\mathbb{C}}\sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}: |t-t_0|\leq\frac{\eta}{\theta_1 L(z+t_0\mathbf{b})}\right\}<+\infty.\end{aligned}$$

Thus $L^*\in Q^n_{\mathbf{b}}$. □

3 Main theorem

Now we prove several propositions that indicate ways of construction of functions from the class $Q^n_{\mathbf{b}}$.

Theorem 1. *If $l\in Q$ and $\inf\{l(t): t\in\mathbb{C}\}=c>0$, then $L\in Q^n_{\mathbf{b}}$, where*

$$L(z)=\frac{1}{c}\Big(1+\sum_{k=1}^{n}\big(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|)\big)\Big)l\Big(\prod_{j=1}^{n}z_j\Big),\quad\text{and}\quad\prod_{j\in\varnothing}(\cdot)=1.$$

Proof. Note that in the definition of $Q^n_{\mathbf{b}}$ it is required that inequality (5) holds for all $\eta>0$. But in view of (1)–(4) function $\lambda_1^{\mathbf{b}}(\eta)$ is nonincreasing and $\lambda_2^{\mathbf{b}}(\eta)$ is nondecreasing. So it is sufficient to require in definition of $Q^n_{\mathbf{b}}$ that inequality (5) is true for all $\eta\geq 1$. Indeed let this inequality holds for $\eta^*>1$. Then for all $\widetilde{\eta}$ such that $0<\widetilde{\eta}<1\leq\eta^*<+\infty$, the following inequalities hold $\lambda_1^{\mathbf{b}}(\widetilde{\eta})\geq\lambda_1^{\mathbf{b}}(\eta^*)>0$, $\lambda_2^{\mathbf{b}}(\widetilde{\eta})\leq\lambda_2^{\mathbf{b}}(\eta^*)<+\infty$. Thus inequality (5) holds for all $\eta>0$. Below we assume that $\eta\geq 1$.

Besides, we suppose that $\inf\{l(t): t\in\mathbb{C}\}=1$. If this infimum does not equal 1, then we can consider the function $\widetilde{l}(t)=\dfrac{l(t)}{\inf\{l(t): t\in\mathbb{C}\}}$, for which this equality holds.

So we consider the case $\eta\geq 1$ and $\inf\{l(t): t\in\mathbb{C}\}=1$. We shall prove that for all $\eta\geq 1$ the following inequalities hold

$$\begin{aligned}\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}&\Bigg\{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt|\prod_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt)\Big)\\ &\Big/\Big(\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\Big):\\ |t-t^0|\leq&\frac{\eta}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\Bigg\}>0\end{aligned} \tag{6}$$

and

$$\sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\Bigg\{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt|\prod_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt)\Big)$$
$$\Big/\Big(\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\Big):$$
$$|t-t^0|\le\frac{\eta}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\Bigg\}<\infty. \tag{7}$$

For this end we use the fact that $l\in Q$. According to our choice $\inf\{l(t):t\in\mathbb{C}\}=1$ and

$$\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\ge 1.$$

Hence, we obtain that

$$|t-t^0|\le\frac{\eta}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\le\eta. \tag{8}$$

It remains to estimate the module

$$\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|=\Big|\Big(\prod_{j=1}^{n}(z_j+b_jt)-(z_1+b_1t^0)\prod_{j=2}^{n}(z_j+b_jt)\Big)$$
$$+\Big((z_1+b_1t^0)\prod_{j=2}^{n}(z_j+b_jt)-\prod_{j=1}^{2}(z_j+b_jt^0)\prod_{j=3}^{n}(z_j+b_jt)\Big)+\cdots$$
$$+\Big(\prod_{j=1}^{k-1}(z_j+b_jt^0)\prod_{j=k}^{n}(z_j+b_jt)-\prod_{j=1}^{k}(z_j+b_jt^0)\prod_{j=k+1}^{n}(z_j+b_jt)\Big)+\cdots \tag{9}$$
$$+\Big((z_j+b_nt)\prod_{j=1}^{n-1}(z_j+b_jt^0)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\Big|.$$

We estimate each of obtained n differences separately. In particular n-th difference can be estimated as

$$\Big|(z_j+b_nt)\prod_{j=1}^{n-1}(z_j+b_jt^0)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|=\prod_{j=1}^{n-1}|z_j+b_jt_0|\,|b_n|\,|t-t_0|$$
$$\le\frac{\eta\prod_{j=1}^{n-1}|z_j+b_jt_0|\,|b_n|}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\cdot\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.$$

Applying the inequality (8) and using that $\eta>1$, $(n-1)$-th differences can be estimated as

$$\Big|\prod_{j=1}^{n-2}(z_j+b_jt^0)\prod_{j=n-1}^{n}(z_j+b_jt)-\prod_{j=1}^{n-1}(z_j+b_jt^0)\,(z_j+b_jt)\Big|=\prod_{j=1}^{n-2}|z_j+b_jt^0|\,|b_{n-1}|\,|t-t^0|\,|z_n+b_nt|$$
$$=\prod_{j=1}^{n-2}|z_j+b_jt^0|\,|b_{n-1}|\,|t-t^0|\,|z_n+b_nt^0+b_n(t-t^0)|$$
$$\le\prod_{\substack{j=1\\ j\ne n-1}}^{n}|z_j+b_jt^0|\,|b_{n-1}|\,|t-t^0|+\prod_{j=1}^{n-2}|z_j+b_jt^0|\,|b_{n-1}|\,|b_n|\,|t-t^0|^2$$
$$\le\frac{\eta\prod\limits_{j=1,j\ne n-1}^{n}|z_j+b_jt_0|\,|b_{n-1}|}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}$$
$$+\frac{\eta^2\prod\limits_{j=1}^{n-2}|z_j+b_jt_0|\,|b_{n-1}|\,|b_n|}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}$$
$$\le\frac{\eta^2\Big(\prod\limits_{j=1,j\ne n-1}^{n}|z_j+b_jt_0|\,|b_{n-1}|+\prod\limits_{j=1}^{n-2}|z_j+b_jt_0|\,|b_{n-1}|\,|b_n|\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\cdot\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.$$

For arbitrary k-th difference, $1\le k\le n$, of (9) we can obtain estimate

$$\Big|\prod_{j=1}^{k-1}(z_j+b_jt^0)\prod_{j=k}^{n}(z_j+b_jt)-\prod_{j=1}^{k}(z_j+b_jt^0)\prod_{j=k+1}^{n}(z_j+b_jt)\Big|$$
$$=\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}|z_j+b_jt|\,|b_k|\,|t-t_0|$$
$$=\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}|z_j+b_jt^0+b_j(t-t^0)|\,|b_k|\,|t-t_0|$$
$$\le\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|\,|t-t^0|)|b_k|\,|t-t_0|$$
$$\le\frac{\eta\,|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|\eta)}{\Big(1+\prod\limits_{j=1}^{n}(|z_j+b_jt^0|+|b_j|)-\prod\limits_{j=1}^{n}|z_j+b_jt^0|\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}$$
$$\le\frac{\eta^{n-k}|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)}{\Big(1+\prod\limits_{j=1}^{n}(|z_j+b_jt^0|+|b_j|)-\prod_{j=1}^{n}|z_j+b_jt^0|\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.$$

Thus, returning to (9) and considering that $\eta^j\le\eta^n$ for all j, $1\le j\le n$, we obtain the following inequality

$$
\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|
$$

$$
\le \sum_{k=1}^{n}\frac{\eta^{n-k}|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}
$$

$$
\le \eta^n\sum_{k=1}^{n}\frac{|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}
\times\frac{1}{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}
$$

$$
\le \frac{\eta^n\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}
\le \frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.
$$

Then for all $\eta\ge 1$

$$
\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\left\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}:\right.
$$

$$
\left.|t-t^0|\le\frac{\eta}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\cdot\frac{1}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\right\}
$$

$$
\ge \inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\left\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}:|t-t^0|\le\eta\right\}
$$

$$
\times\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\left\{\frac{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}:\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|\le\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\right\}. \tag{10}
$$

The first factor in the obtained inequality is a fractional rational expression with the same

degrees of the numerator and denominator by variable z_j, and by t, t^0, respectively. Thus the corresponding infimum is not equal to zero. Suppose that the second expression equals zero.

Then there exists sequences (z^p), (t^0_p), for which

$$
\inf_{t}\left\{\frac{l\Big(\prod\limits_{j=1}^{n}(z_j^p+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j^p+b_jt^0_p)\Big)}:\Big|\prod_{j=1}^{n}(z_j^p+b_jt)-\prod_{j=1}^{n}(z_j^p+b_jt^0_p)\Big|\le\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0_p)\Big)}\right\}\underset{p\to+\infty}{\to}0.
$$

Denoting $u_p(t)=\prod\limits_{j=1}^{n}(z_j^p+b_jt)$, and $v_p(t^0_p)=\prod\limits_{j=1}^{n}(z_j^p+b_jt^0_p)$, we obtain that

$$
\inf_{t}\left\{\frac{l(u_p(t))}{l(v_p(t^0_p))}:|u_p(t)-v_p(t^0_p)|\le\frac{\eta}{l(v_p(t^0_p))}\right\}\underset{p\to+\infty}{\to}0.
$$

But

$$
\inf_{t}\left\{\frac{l(u_p(t))}{l(v_p(t^0_p))}:|u_p(t)-v_p(t^0_p)|\le\frac{\eta}{l(v_p(t^0_p))}\right\}\ge\inf_{u}\left\{\frac{l(u)}{l(v_p(t^0_p))}:|u-v_p(t^p_0)|\le\frac{\eta}{l(v_p(t^0_p))}\right\},
$$

and $\inf\limits_{v\in\mathbb{C}}\inf\limits_{u}\left\{\frac{l(u)}{l(v)}:|u-v|\le\frac{\eta}{l(v)}\right\}=0$, that contradicts the condition $l\in Q$. Thus, the second factor in (10) is also positive, so the inequality (6) is correct.

Using similar considerations, we can prove the similar inequality for sup. Indeed, for all $\eta\ge1$ the following inequalities hold

$$
\sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\left\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}:\right.
$$

$$
\left.|t-t^0|\le\frac{\eta}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\frac{1}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\right\}
$$

$$
\le\sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\left\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}:\right.
$$

$$
\left.|t-t^0|\le\frac{\eta}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\right\}
$$

$$
\times\sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\left\{\frac{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}:\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|\le\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\right\} \tag{11}
$$

$$\leq \sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\left\{\frac{\left(1+\sum\limits_{k=1}^{n}\left(|b_k|\prod\limits_{j=1}^{k-1}|z_j+|!b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\right)\right)}{\left(1+\sum\limits_{k=1}^{n}\left(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\right)\right)}:|t-t^0|\leq\eta\right\}.$$

As above for infimum in the first brackets we obtain a fractional rational expression with the same degrees of the numerator and denominator by z_j, and by t, t^0 respectively. Hence corresponding supremum does not equal infinity. Suppose that the second expression is equal to infinity. Then there exist (z^p), (t_p^0) with property

$$\sup_t\left\{\frac{l\Big(\prod\limits_{j=1}^{n}(z_j^p+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j^p+b_jt_p^0)\Big)}:\Big|\prod_{j=1}^{n}(z_j^p+b_jt)-\prod_{j=1}^{n}(z_j^p+b_jt_p^0)\Big|\leq\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt_p^0)\Big)}\right\}\underset{p\to+\infty}{\to}\infty.$$

Denoting $u_p(t)=\prod\limits_{j=1}^{n}(z_j^p+b_jt)$, and $v_p(t_p^0)=\prod\limits_{j=1}^{n}(z_j^p+b_jt_p^0)$, we obtain

$$\sup_t\left\{\frac{l(u_p(t))}{l(v_p(t_p^0))}:|u_p(t)-v_p(t_p^0)|\leq\frac{\eta}{l(v_p(t_p^0))}\right\}\underset{p\to+\infty}{\to}\infty.$$

But

$$\sup_t\left\{\frac{l(u_p(t))}{l(v_p(t_p^0))}:|u_p(t)-v_p(t_p^0)|\leq\frac{\eta}{l(v_p(t_p^0))}\right\}\leq\sup_u\left\{\frac{l(u)}{l(v_p(t_p^0))}:|u-v_p(t_0^p)|\leq\frac{\eta}{l(v_p(t_p^0))}\right\},$$

and $\sup_{v\in\mathbb{C}}\sup_u\left\{\frac{l(u)}{l(v)}:|u-v|\leq\frac{\eta}{l(v)}\right\}=\infty$, that contradicts the condition $l\in Q$. Thus, the second factor in (11) is also positive, so the inequality (7) is valid. Hence, we deduce that the function

$$\frac{1}{c}\Big(1+\sum_{k=1}^{n}(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|))\Big)l\Big(\prod_{j=1}^{n}z_j\Big)$$

belongs to the class $Q_{\mathbf{b}}^n$. □

4 REMARKS TO MAIN THEOREM

Remark 1. *The condition* $\inf\{l(t):t\in\mathbb{C}\}=c>0$ *is not essential. In fact, every function* $l\in Q$*, which satisfies the equality* $\inf\{l(t):t\in\mathbb{C}\}=0$*, can be replaced by the function* $l(t)+1$*, which also belongs to the class* Q*.*

Proof. Indeed, for the positive continuous function $l(t)$ the inequality holds

$$\frac{l(t)}{l(t_0)}\leq\frac{l(t)+1}{l(t_0)+1}<\frac{l(t)}{l(t_0)}+1, \tag{12}$$

where the right part is true for all $t,t_0\in\mathbb{C}$, and the left part is true for all $t,t_0\in\mathbb{C}$ such that $l(t)\leq l(t_0)$. The right inequality is equivalent to the following

$$l(t_0)(l(t)+1)<(l(t)+l(t_0))(l(t_0)+1)\quad\text{or}\quad l(t_0)l(t)+l(t_0)<l(t)l(t_0)+l^2(t_0)+l(t)+l(t_0),$$

i. e. $0<l^2(t_0)+l(t)$. But this inequality holds for the function $l(t)$ for all $t,t_0\in\mathbb{C}$.

From the left part we similarly obtain $l(t)l(t_0)+l(t)\leq l(t_0)(l(t)+1)$. Hence $l(t)\leq l(t_0)$.

Evaluating the supremum for the right part of inequality (12) and the infimum for the left side and using that $l(t)\in Q$, we obtain

$$\begin{aligned}0<\inf\left\{\frac{l(t)}{l(t_0)}:|t-t_0|\leq\frac{\eta}{l(t_0)},t\in\mathbb{C}\right\}&\leq\inf\left\{\frac{l(t)}{l(t_0)}:|t-t_0|\leq\frac{\eta}{l(t_0)+1},t\in\mathbb{C}\right\}\\&\leq\inf\left\{\frac{l(t)+1}{l(t_0)+1}:|t-t_0|\leq\frac{\eta}{l(t_0)+1},t\in\mathbb{C}\right\}\\&\leq\sup\left\{\frac{l(t)+1}{l(t_0)+1}:|t-t_0|\leq\frac{\eta}{l(t_0)+1},t\in\mathbb{C}\right\}\\&\leq\sup\left\{\frac{l(t)}{l(t_0)}+1:|t-t_0|\leq\frac{\eta}{l(t_0)+1},t\in\mathbb{C}\right\}\\&\leq\sup\left\{\frac{l(t)}{l(t_0)}+1:|t-t_0|\leq\frac{\eta}{l(t_0)},t\in\mathbb{C}\right\}<\infty.\end{aligned}$$

These inequalities imply $l(t)+1\in Q$. □

Remark 2. *In fact, analysis of the proof of Theorem 1 indicates that we can somehow decrease function L. For each* $\mathbf{b}=(b_1,b_2,\ldots,b_n)\in\mathbb{C}^n$*, such that* $\prod_{j=1}^{n}|b_j|\neq0$*,* $l\in Q$ *and* $\inf\{l(t):t\in\mathbb{C}\}=c>0$*, we have* $L\in Q_{\mathbf{b}}^n$*, where*

$$L(z)=\frac{1}{c}\Big(\sum_{k=1}^{n}(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|))\Big).$$

The appearance of term 1 in the proof of Theorem 1 is necessary for lower estimate of the function $\Big(\sum\limits_{k=1}^{n}(|b_k|\prod\limits_{j=1}^{k-1}|z_j|\prod\limits_{j=k+1}^{n}(|z_j|+|b_j|))\Big)^j$, where $j=1,2,\ldots,n$. We can take the direction $\widetilde{\mathbf{b}}=\mathbf{b}\Big/\prod\limits_{j=1}^{n}|b_j|$ instead of $\mathbf{b}$ under the previous condition $\prod\limits_{j=1}^{n}|b_j|\neq0$, because by Lemma 1 the function L belongs to the class $Q_{\theta\mathbf{b}}^n$, with $\theta=\frac{1}{\prod_{j=1}^{n}|b_j|}$.

Then all considerations of previous theorem should be repeated, omitting the term 1 in the appropriate places. Alternatively we can take a larger function.

Remark 3. *If* $l^*\in Q$*,* $l\in Q$*,* $\inf\{l(t):t\in\mathbb{C}\}=c>0$*, and for all* $z\in\mathbb{C}^n$ *the following inequalities hold*

$$l^*\Big(\prod_{j=1}^{n}z_j\Big)\geq c_1\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|)\Big)\Big)$$

and

$$l^*\Big(\prod_{j=1}^{n}z_j\Big)\leq c_2\Big(\prod_{j=1}^{n}(|z_j|+|b_j|)-\prod_{j=1}^{n}|z_j|\Big),$$

then $L\in Q_{\mathbf{b}}^n$*, where* $L(z)=\frac{1}{c}l^*\Big(\prod\limits_{j=1}^{n}z_j\Big)l\Big(\prod\limits_{j=1}^{n}z_j\Big)$, $\mathbf{b}=(b_1,b_2,\ldots,b_n)$, $c_1>0,c_2>0$.

Proof. Without loss of generality, we may suppose $\inf\{l(t) : t \in \mathbb{C}\} = 1$ as in Theorem 1. Then we can repeat the considerations of this theorem, taking everywhere the function $l^*\Big(\prod\limits_{j=1}^{n} z_j\Big)$ instead of

$$1 + \sum_{k=1}^{n} \Big(|b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|)\Big).$$

Therefore we obtain

$$\Big|\prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0)\Big| \leq \eta^n \frac{\sum\limits_{k=1}^{n} \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t^0| \prod\limits_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j|)\Big)}{\min\{1, c_1^n\} l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)}$$
$$\leq \frac{\eta^n}{\min\{c_1, c_1^{n+1}\} l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)}.$$

Denoting $\widetilde{c} = \min\{c_1, c_1^{n+1}\}$, for all $\eta \geq 1$ we obtain the following inequality

$$\inf_{z\in\mathbb{C}^n} \inf_{t^0\in\mathbb{C}} \inf_{t} \left\{ \frac{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : |t - t^0| \leq \frac{\eta}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\}$$
$$\geq \inf_{z\in\mathbb{C}^n} \inf_{t^0\in\mathbb{C}} \inf_{t} \left\{ \frac{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : \Big|\prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0)\Big| \leq \frac{\eta^n}{\widetilde{c}\, l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\}$$
$$\times \inf_{z\in\mathbb{C}^n} \inf_{t^0\in\mathbb{C}} \inf_{t} \left\{ \frac{l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : \Big|\prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0)\Big| \leq \frac{\eta^n}{\widetilde{c}\, l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\}. \quad (13)$$

Since $l(t) \in Q$, by similar considerations as in Theorem 1 it can be showed that the product in (13) is greater than zero. It is obviously that we can prove

$$\sup_{z\in\mathbb{C}^n} \sup_{t^0\in\mathbb{C}} \sup_{t} \left\{ \frac{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : |t - t^0| \leq \frac{\eta}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\} < \infty. \quad (14)$$

In view of (13), (14) we obtain that the function $l^*\Big(\prod\limits_{j=1}^{n} z_j\Big) l\Big(\prod\limits_{j=1}^{n} z_j\Big)$ belongs to the class $Q^n_{\mathbf{b}}$. □

Remark 4. *We can take the following functions*

$$\prod_{j=1}^{n}(|z_j| + |b_j|) - \prod_{j=1}^{n}|z_j| \qquad \text{or} \qquad \sum_{k=1}^{n}\Big(|b_k| \prod_{\substack{j=1\\ j\neq k}}^{n}(|z_j| + |b_j|)\Big)$$

instead of the expression $\sum\limits_{k=1}^{n} \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j| \prod\limits_{j=k+1}^{n} (|z_j| + |b_j|)\Big)$ *in Theorem 1.*

It follows from Lemma 5 and notion

$$1+\prod_{j=1}^{n}(|z_j|+|b_j|)-\prod_{j=1}^{n}|z_j| \asymp 1+\sum_{k=1}^{n}\Big(|b_k|\prod_{\substack{j=1\\ j\neq k}}^{n}(|z_j|+|b_j|)\Big) \asymp 1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j| \prod_{j=k+1}^{n}(|z_j|+|\mathbf{b}_j|)\Big).$$

Proposition 1. *If $L \in Q^n_{\mathbf{b}}$, then for every $z^0 \in \mathbb{C}^n$ we have $l_{z^0} \in Q$ $(l_{z^0}(t) \equiv L(z^0 + t\mathbf{b}))$.*

Proof. We remark that (1)–(5) imply for every $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$

$$\forall \eta > 0 \;\; 0 < \lambda_1^{\mathbf{b}}(z,\eta) \leq \lambda_1^{\mathbf{b}}(z,t_0,\eta) \leq 1 \leq \lambda_2^{\mathbf{b}}(z,t_0,\eta) \leq \lambda_2^{\mathbf{b}}(z,\eta) < +\infty.$$

These inequalities imply that $l_{z^0} \in Q$. □

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Бандура А.І. *Властивості додатних неперервних функцій у $\mathbb{C}^n$* // Карпатські матем. публ. — 2015. — Т.7, №2. — C. 137–147.

Досліджено властивості класів $Q^n_{\mathbf{b}}$ та Q додатних непервних функцій. Доведено, що деякі композиції функцій із класу Q належать класу $Q^n_{\mathbf{b}}$. Встановлено зв'язок між функціями цих класів.

Ключові слова і фрази: додатна функція, неперервна функція, декілька комплексних змінних.

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BODNAR D.I., BUBNIAK M.M.

ON CONVERGENCE OF $(2,1,\dots,1)$-PERIODIC BRANCHED CONTINUED FRACTION OF THE SPECIAL FORM

$(2,1,\dots,1)$-periodic branched continued fraction of the special form is defined. Conditions of convergence are established for 2-periodic continued fraction and $(2,1,\dots,1)$-periodic branched continued fraction of the special form. Truncation error bounds are estimated for these fractions under additional conditions.

Key words and phrases: periodic branched continued fractions of the special form, convergence.

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INTRODUCTION

Periodic continued fractions are an important subclass of continued fractions

$$b_0+\mathop{\mathrm{D}}_{k=1}^{\infty}\frac{a_k}{b_k}=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots}}=b_0+\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\dots, \tag{1}$$

where $a_i, b_0, b_i \in \mathbb{C}; i \geq 1$. A fraction (1) is called p-periodic, if its elements satisfy the following conditions: $a_{pn+k} = a_k$ and $b_{pn+k} = b_k$; $n \geq 0$; $1 \leq k \leq p$; $p \in \mathbb{N}$. L. Euler, D. Bernoulli, E. Kahl, E. Galios, A. Pringsheim, W. Leighton, O. Perron, R. Lane, H. Wall, W. Jones, W. Thron, H. Waadeland, L. Loretzen, A. F. Beardon etc. investigated p-periodic fractions. The reviews of corresponding results can be found in [5–7]. It is known (see [5, p. 181]), that the set

$$\Omega=\{z\in\mathbb{C}:|\arg(z+1/4)|<\pi\} \tag{2}$$

is the convergence set of the 1-periodic continued fraction

$$1+\frac{c|}{|1}+\frac{c|}{|1}+\dots. \tag{3}$$

Moreover, attracting and repelling fixed points of the linear fractional transformation $t(\omega) = 1 + c/\omega$ are the points

$$x=(1+\sqrt{1+4c})/2,\quad y=(1-\sqrt{1+4c})/2. \tag{4}$$

In [5, p. 49] it is proved that the the following relations are valid for the fraction (3)

$$P_n=\frac{x^{n+2}-y^{n+2}}{x-y},\quad Q_n=\frac{x^{n+1}-y^{n+1}}{x-y},\quad n\geq 1,\quad \lim_{n\to\infty}\frac{P_n}{Q_n}=x. \tag{5}$$

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1 MAIN RESULTS

We consider the branched continued fraction (BCF) of the special form

$$1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{1}, \tag{6}$$

where $a_{i(k)} \in \mathbb{C}$, $i(k) \in \mathcal{I}$, $\mathcal{I}$ is a set of multiindex, $\mathcal{I} = \{i_1 i_2 \dots i_k : 1 \leq i_k \leq i_{k-1}; k \geq 1; i_0 = N\}$, N is a fixed natural number. Some results according to these BCF are in [3,4].

Continued fraction

$$1+a_{i(m)}\left(1+\mathop{\mathrm{D}}_{q=1}^{\infty}\frac{a_{i(m+q)}}{1}\right)^{-1},$$

where $i(m) \in \mathcal{I}$, $i_m = i_{m+q} = r$, $q \geq 1$, is called the $i(m)$-th branch of the r-th order of BCF (6).

Definition. *A fraction (6) is called $\overrightarrow{p}$-periodic branched continued fraction of the special form, where $\overrightarrow{p} = (p_1, p_2, \dots, p_N)$, $p_j \in \mathbb{N}$, $j = \overline{1,N}$, if all $i(m)$-th branches are the identical p_{i_m}-periodic continued fraction for each fixed i_m.*

Let BCF (6) be a $\overrightarrow{p}$-periodic fraction. Than its elements satisfy the following conditions

$$a_{\underbrace{r\dots r}_{q}}=a_{\underbrace{r\dots r}_{s}}\quad or \quad a_{i(m)\underbrace{r\dots r}_{q}}=a_{\underbrace{r\dots r}_{s}}, \tag{7}$$

where $q \geq 1$; $q = n \cdot p_r + s$; $r = \overline{1,N}$; $s = \overline{1,p_r}$; $m \geq 1$; $i(m) \in \mathcal{I}$; $r < i_m$; $n \geq 0$. Each $i(m)$-th branch of the r-th order is called the r-th branch of such fraction.

We introduce the notation $a_{\underbrace{r\dots r}_{s}} = c_{r,s}$ for elements of the fraction (6). Then $\overrightarrow{p}$-periodic BCF can be written as follows

$$1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{c_{i_k,s}}{1}. \tag{8}$$

We investigate the convergence of $(2,1,\dots,1)$-periodic BCF with N branches

$$1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\ddots}}}}+\cfrac{c_{2,1}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\cfrac{c_{1,1}}{1+\ddots}}}+\cfrac{c_{2,1}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\ddots}}+\cfrac{c_{2,1}}{1+\ddots+\ddots}}}$$
$$+\dots+\cfrac{c_{N,1}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\ddots}}+\cfrac{c_{2,1}}{1+\cfrac{c_{1,1}}{1+\ddots}+\cfrac{c_{2,1}}{1+\ddots}}+\dots+\cfrac{c_{N,1}}{1+\cfrac{c_{1,1}}{1+\ddots}+\cfrac{c_{2,1}}{1+\ddots}+\dots+\cfrac{c_{N,1}}{1+\ddots}}}. \tag{9}$$

For this we define tails of $\overrightarrow{p}$-periodic BCF (8) with initial conditions: $R_0^{(q,j)} = 1$, $q = \overline{1,N}$, $1 \leq j \leq n$, $n \geq 1$, and the recurrence relations

$$\begin{cases} R_n^{(1,s)}=1+\dfrac{c_{1,s}}{R_{n-1}^{(1,s+1)}}, & 1\leq s\leq p_1,\\ R_n^{(q,s)}=1+\sum_{k=1}^{q-1}\dfrac{c_{k,1}}{R_{n-1}^{(k,2)}}+\dfrac{c_{q,s}}{R_{n-1}^{(q,s+1)}}, & q=\overline{1,N}; 1\leq s\leq p_q, \end{cases} \tag{10}$$

where $n \geq 1$ $p_q \in \mathbb{N}$, $q = \overline{1,N}$. Then $R_n^{(q,j)} = R_n^{(q,s)}$ and $R_n^{(q,m)} = R_n^{(q-1,1)} + c_{q,m}/R_{n-1}^{(q,m+1)}$, $n \geq 1, q = \overline{1,N}$ $1 \leq j \leq n, 1 \leq m \leq p_q - 1$, $p_q \in \mathbb{N}$.

Thus, the n-th approximants of BCF (8) are equal to $F_n = R_n^{(N,1)}$, $n \geq 1$, $F_0 = 1$.

For investigation of truncation error bounds of the fraction (8) we have used a formula for $n \geq 0, m > 0$, that had been proved in [1], such as

$$F_{n+m} - F_n = \sum_{\vec{k} \in \mathcal{I}_{n+1}^{(N)}} \frac{c_{1,1}^{k_{1,1}} c_{1,2}^{k_{1,2}} c_{2,1}^{k_{2,1}} \cdots c_{N,1}^{k_{N,1}}}{\prod_{j=1}^{k_1} \left(R_{m+l_1-j}^{(1,j+1)} \cdot R_{l_1-j}^{(1,j+1)}\right) \cdots \prod_{j=1}^{k_N} \left(R_{m+n-j}^{(N,1)} \cdot R_{n-j}^{(N,1)}\right)}, \tag{11}$$

where $\mathcal{I}_{n+1}^{(N)} := \{ \vec{k} = (k_1, k_2, \ldots, k_N) : k_1 = k_{1,1} + k_{1,2}; k_l \geq 0; l = \overline{1,N}; \sum_{l=1}^N k_l = n+1 \}$, $l_i = n - \sum_{t=i+1}^N k_t$, $R_{-1}^{(q,j)} = 1$, $q = \overline{1,N}$, $j = \overline{1,p_q}$, $p_1 = 2$, $p_2 = \ldots = p_N = 1$, $k_{r,s}$ is defined in [3]. Now we consider the 2-periodic continued fraction

$$1 + \frac{a|}{|1} + \frac{b|}{|1} + \frac{a|}{|1} + \frac{b|}{|1} + \ldots. \tag{12}$$

Let $\lambda = 1 + a + b$ and $\lambda \neq 0$. According to [5, Theorems 2.19, 2.20], [6, Theorem 1.6] we have that the even part and the odd part of the fraction (12) are equal to

$$1 + a\left(1 + b + \mathop{D}_{k=2}^{\infty} \frac{c_k}{d_k}\right)^{-1}, \qquad 1 + a + \mathop{D}_{k=1}^{\infty} \frac{c_k}{d_k}$$

respectively, where $c_k = -ab$, $d_k = 1 + a + b$. Next, let P_ν, Q_ν be the ν-th nominator and the ν-th denominator of 1-periodic continued fraction

$$1 + \frac{-ab/(1+a+b)^2|}{|1} + \frac{-ab/(1+a+b)^2|}{|1} + \ldots, \quad \nu \geq 1. \tag{13}$$

Then, according to formulas (5), we have for $k \geq 0$

$$P_k = \frac{\widetilde{x}^{k+2} - \widetilde{y}^{k+2}}{\widetilde{x} - \widetilde{y}}, \quad Q_k = \frac{\widetilde{x}^{k+1} - \widetilde{y}^{k+1}}{\widetilde{x} - \widetilde{y}},$$

where $\widetilde{x} = (1 + \sqrt{1 - 4ab/\lambda^2})/2$, $\widetilde{y} = (1 - \sqrt{1 - 4ab/\lambda^2})/2$.

Let $f_n^{(s)} = 1 + \mathop{D}_{k=s}^{n+s-1} \frac{a_k}{1}$ be the s-th tail of the fraction (12), $n \geq 1, 1 \leq s \leq n$, where $a_{2k-1} = a$, $a_{2k} = b$, $k \geq 1$. Then the following formulas

$$f_{2\nu}^{(j)} = \frac{\lambda P_{\nu-1}}{-a_j Q_{\nu-1} + \lambda P_{\nu-1}}, \quad \nu > 0, \quad f_{2\nu+1}^{(j)} = \frac{-a_{j+1} Q_\nu + \lambda P_\nu}{Q_\nu}, \quad \nu \geq 0, \quad j = 1, 2,$$

are valid for the 1-st and 2-nd tails of 2-periodic continued fraction (12).

Lemma. *Let the elements of 2-periodic fraction (12) satisfy the condition $-ab/\lambda^2 \in \Omega$, where $\lambda = 1 + a + b$, $\lambda \neq 0$, and Ω is defined by formula (2). Then:*

1. *the fraction (12) converges to value $x = (1 + a - b + \lambda\sqrt{1 - 4ab/\lambda^2})/2$;*

2. *if $f_{2k+1}^{(j)} \neq 0$, $k \geq 0$, $j = \overline{1,2}$, and $|-a + \lambda P_k/Q_k| \geq \varepsilon_1 > 0$, $k \geq 0$, then truncation error bounds are valid*

$$|f_n - x| \leq Cq^{[(n+1)/2]}, \quad n \geq 0, \tag{14}$$

where $q = \left|\frac{1 - \sqrt{1 - 4ab/\lambda^2}}{1 + \sqrt{1 - 4ab/\lambda^2}}\right| < 1$, $C = \frac{|\widetilde{x}||\lambda|(1+q)^2}{(1-q)^2} \max\left\{\frac{1}{\varepsilon_2}; \frac{|a|}{\varepsilon_1^2}\right\}$ $\varepsilon_2 = |b| + |\lambda| M$, $M = |\widetilde{x}|(1+q^2)/(1-q)$, $\widetilde{x} = (1 + \sqrt{1 - 4ab/\lambda^2})/2$.

Proof. Let $c = -ab/\lambda^2$. Since $c \in \Omega$, then 1-periodic continued fraction (13) converges and its value is $\widetilde{x}$, moreover $|\widetilde{x}| > |\widetilde{y}|$. Next, since $\lambda \neq 0$, then $\lim_{\nu\to\infty} f_{2\nu+1} = \lim_{\nu\to\infty} f_{2\nu} = x$. From this it follows that the fraction (12) converges and $\lim_{n\to\infty} f_n = x$.

Since $c \in \Omega$, all approximants of the fraction (13) are not equal to zero. It follows that $f_{2n}^{(j)} \neq 0$, $n \geq 1$, $j = \overline{1,2}$. For $n \geq 1$ and $m \geq 1$ we estimate the difference $|f_{n+2m} - f_n|$, using formula (11). By virtue of $\left|\frac{P_k}{Q_k}\right| = \left|\frac{\widetilde{x}^{k+2} - \widetilde{y}^{k+2}}{\widetilde{x}^{k+1} - \widetilde{y}^{k+1}}\right| = |\widetilde{x}| \left|\frac{1 - (\widetilde{y}/\widetilde{x})^{k+2}}{1 - (\widetilde{y}/\widetilde{x})^{k+1}}\right|$ for $k \geq 0$ the following inequalities are valid $\mu \leq |P_k/Q_k| \leq M$, where $\mu = |\widetilde{x}|(1-q)$, and $|-b + \lambda P_k/Q_k| \leq \varepsilon_2$.

Let n and k be arbitrary natural numbers, moreover $n = 2r+1$, $k = r+m$, $r \geq 0$, $m \geq 0$. Then

$$|f_{2k+1} - f_{2r+1}| = \frac{|a|^{r+1}|b|^{r+1}}{\prod_{q=1}^{r+1}\left(|f_{2k-2q+2}^{(2)}||f_{2k-2q+1}^{(1)}||f_{2r+1-2q+1}^{(2)}||f_{2r+1-2q}^{(1)}|\right)},$$

$$\prod_{q=1}^{r+1}\left|f_{2(k-q+1)}^{(2)} f_{2(k-q)+1}^{(1)}\right| = |\lambda|^{r+1}\left|\frac{P_{k-1}}{-bQ_{k-r} + \lambda P_{k-r}}\right| \geq |\lambda|^{r+1}|\widetilde{x}|^r \frac{(1-q)}{(1+q)\varepsilon_2},$$

$$\prod_{q=1}^{r+1}\left|f_{2(r-q+1)}^{(2)} f_{2(r-q)+1}^{(1)}\right| = |\lambda|^r |P_{r-1}| \geq |\lambda|^r |\widetilde{x}|^r \frac{1-q}{1+q}.$$

From this, we have

$$|f_{2k+1} - f_{2r+1}| \leq \frac{(ab/\lambda^2)^{r+1}|\lambda|(1-q)^2}{|x|^{2r}(1+q)^2 M} = \frac{|\widetilde{y}|^{r+1}|\widetilde{x}|^{r+1}|\lambda|(1+q)^2}{|\widetilde{x}|^{2r+1}(1-q)^2 M} = C_1 \left|\frac{\widetilde{y}}{\widetilde{x}}\right|^{r+1},$$

where $C_1 = |\lambda||x|(1+q)^2/((1-q)^2 M)$.

Let n and k be arbitrary natural numbers, moreover $n = 2r+1$, $k = r+m$, $r \geq 0$, $m \geq 0$. Then, by analogy we have

$$|f_{2k} - f_{2r}| \leq \frac{|a|^{r+1}|b|^r(1+q)^2}{|\lambda|^{2r}|\widetilde{x}|^{2r-1}(1-q)^2\varepsilon_1^2} = C_2\left|\frac{\widetilde{y}}{\widetilde{x}}\right|^r, \quad C_2 = \frac{|a||\widetilde{x}|(1+q)^2}{(1-q)^2\varepsilon_1^2}.$$

Finally, we obtain truncation error bounds (14) for $m \to \infty$. □

Now we consider the linear fractional transformation

$$t_1(\omega) = 1 + \cfrac{c_{1,1}}{1 + \cfrac{c_{1,2}}{\omega}}. \tag{15}$$

Let X_1 be the attracting fixed point of this transformation, X_j, Y_j be the attracting and repelling fixed points of $t_j(\omega) = X_{j-1} + c_{j,1}/\omega$, $j = \overline{2,N}$. It is known in [5, p. 190], that

$$X_1 = \left(\lambda - 2c_{1,2} + \lambda\sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2}\right)/2. \tag{16}$$

Theorem. *Let $\mu = -c_{1,1}c_{1,2}/\lambda^2$, $\lambda = 1 + c_{1,1} + c_{1,2}$, $\lambda \neq 0$, $\mu \in \Omega_1$, where Ω_1 is defined by the formula (2), and let the elements of the fraction (9) satisfy the following conditions $c_{j,1} \in \Omega_j$, $j = \overline{2,N}$, where $\Omega_j = \{z \in \mathbb{C} : |\arg(z + X_{j-1}^2/4)| < \pi\}$. Then:*

1. *the fraction (9) converges and its value is $F = X_N$;*
2. *moreover, if $R_{2n+1}^{(j,1)} \neq 0$, $n \geq 0$, $j = 1,2$; $|-c_{1,1} + \lambda P_k/Q_k| \geq \varepsilon_1 > 0$, $k \geq 1$,*

$$|c_{j,1}| < \frac{1}{4}\prod_{k=1}^{j-1} r_p, \quad j = \overline{2,N}, \tag{17}$$

where $r_1 = |\lambda||\widetilde{x}|\dfrac{(1-\rho_1)\varepsilon_1}{(1+\rho_1)\varepsilon_2}$, $r_k = \nu_k^2$, $\nu_k = (1+d_k)/2$, $d_k = \sqrt{1 - 4|c_{k,1}|/\prod_m^{k-1} r_m}$, $k = \overline{2,N}$, $\widetilde{x} = (1 + \sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2})/2$, $\varepsilon_2 = |c_{1,2}| + |\lambda||\widetilde{x}|(1+\rho_1^2)/(1-\rho_1)$, then for $n \geq 1$ the truncation error bounds are valid

$$|F_n - F| \leq L \cdot \binom{N-1}{N+n-1} \cdot \frac{(\sqrt{\rho_1})^{n+1} - \rho^{n+1}}{\rho_1/\rho - \sqrt{\rho_1}}, \tag{18}$$

where $\rho_1 = \left|\dfrac{1 - \sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2}}{1 + \sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2}}\right|$, $\rho = \max\limits_{j=\overline{2,N}}\{\rho_j\}$, $\rho_j = \dfrac{1}{(1+d_j)^2}$, $L = \prod_{j=1}^N \dfrac{M_j}{\nu_j^4}$, $M_1 = \left(\dfrac{1+\rho_1}{1-\rho_1}\right)^2 \dfrac{\varepsilon_2}{\varepsilon_1\rho_1}$, $M_j = \max\left\{1, \dfrac{|c_{j,1}|}{\rho_j \prod_{m=1}^j \nu_m}\right\}$, $\nu_1 = \min\left\{\varepsilon_1, \dfrac{|\lambda||\widetilde{x}|(1-\rho_1)}{2\varepsilon_2}\right\}$, $j = \overline{2,N}$.

Proof. By induction on q we prove the convergence of the sequence $\{R_n^{(q,1)}\}_{n=1}^\infty$, $q = \overline{1,N}$.

For $q = 1$ the convergence of the sequence $\{R_n^{(1,1)}\}_{n=1}^\infty$ follows from Lemma, i.e. $\lim\limits_{n\to\infty} R_n^{(1,1)} = X_1$, where X_1 is defined by the formula (16). By induction hypothesis the following relations $\lim_{n\to\infty} R_n^{(k,1)} = X_k$, $X_k \neq 0$, $Y_k \neq 0$, hold for $q = k$, where $2 \leq k \leq N-1$. We write $R_n^{(q,1)}$ for $q = k+1$ and for the arbitrary natural n as follows

$$R_n^{(k+1,1)} = R_n^{(k,1)} + \frac{c_{k+1,1}|}{|R_{n-1}^{(k,1)}} + \ldots + \frac{c_{k+1,1}|}{|R_0^{(k,1)}}.$$

Since $c_{k+1,1} \in \Omega_{k+1}$, the linear fractional transformation $\widehat{t}_{k+1}(\omega) = \dfrac{c_{k+1,1}/X_k^2}{1+\omega}$ is loxodromic and from (C) of the [5, Theorem 4.13] we have, that $\lim\limits_{n\to\infty} R_n^{(k+1,1)} = X_{k+1}$, where $X_{k+1} = -y_{k+1}$ and y_{k+1} is the repelling fixed point of $\widehat{t}_{k+1}(\omega)$. Next, since $c_{k+1,1} \neq 0$, then $X_{k+1} \neq 0$, $Y_{k+1} \neq 0$. Hence, $\lim\limits_{n\to\infty} F_n = X_N$.

Let k and m be arbitrary integer numbers and $1 \leq k \leq m$, $m \geq 1$, $k = [k_1/2]$, where k_1 is defined by the formula (11). By virtue of $\lambda \neq 0$, $R_n^{(1,2)} = f_n^{(2)}$ and $R_n^{(1,1)} = f_n^{(1)}$, $n \geq 1$, we have

$$\prod_{j=1}^k \left|R_{2\nu+1}^{(1,2)}\right| \left|R_{2\nu}^{(1,1)}\right| \geq |\lambda|^k \left|\frac{-c_{1,1}Q_\nu + \lambda P_\nu}{-c_{1,1}Q_{\nu-k} + \lambda P_{\nu-k}}\right| \geq |\lambda|^k |\widetilde{x}|^k \frac{(1-\rho_1)\varepsilon_1}{(1+\rho_1)\varepsilon_2},$$

where $\nu = (m + l_1)/2 - j$ and l_1 is defined by formula (11). If $\nu = (m + 1 - l_1)/2 - j$, then

$$\prod_{j=1}^k \left|R_{2\nu+1}^{(1,2)}\right| \left|R_{2\nu}^{(1,1)}\right| \geq |\lambda|^k \left|\frac{-c_{1,1}Q_\nu + \lambda P_\nu}{-c_{1,1}Q_{\nu-k} + \lambda P_{\nu-k}}\right| \geq |\lambda|^k |\widetilde{x}|^k.$$

Next, we have

$$\prod_{j=1}^k \frac{|c_{1,1}c_{1,2}|}{|R_{2m+l_1-2j+1}^{(1,j+1)} R_{2m+l_1-2j}^{(1,j+1)}||R_{l_1-2j+1}^{(1,j+1)} R_{l_1-2j}^{(1,j+1)}|} \leq \frac{1}{C^2}\prod_{j=1}^k \frac{|c_{1,1}c_{1,2}|/\lambda^2}{|\widetilde{x}|^2} = M_1 \left|\frac{\widetilde{y}}{\widetilde{x}}\right|^k.$$

Moreover, according to Lemma the inequality $|R_n^{(1)}| \geq \nu_1$ holds.

Let n be arbitrary natural number. By induction on q we prove that the following inequalities are valid

$$|R_n^{(q,1)}| \geq \prod_{j=1}^q \nu_j, \quad q = \overline{2,N}. \tag{19}$$

For $q = 2$ we can write the tail $R_n^{(2,1)}$ in the form

$$R_n^{(2,1)} = R_n^{(1,1)} + \frac{c_{2,1}|}{|R_{n-1}^{(1,1)}} + \ldots + \frac{c_{2,1}|}{|R_0^{(1,1)}} = R_n^{(1,1)} h_n^{(2,1)}, \quad n \geq 1,$$

where for $r = 2$ and

$$h_n^{(r,1)} = 1 + \frac{c_{r,1}/R_n^{(r-1,1)} R_{n-1}^{(r-1,1)}|}{|1} + \frac{c_{r,1}/R_{n-1}^{(r-1,1)} R_{n-2}^{(r-1,1)}|}{|1} + \ldots + \frac{c_{r,1}/R_1^{(r-1,1)} R_0^{(r-1,1)}|}{|1}. \tag{20}$$

From [2, Lemma 2] it follows: if elements of the reversed fractions $h_n^{(2,1)}$, $n \geq 1$, satisfy the condition $|a_n| < |a| < 1/4$, then the inequality $|h_n^{(2,1)}| \geq \nu_2$ holds. From this we have $\left|\dfrac{c_{2,1}}{R_n^{(1,1)} R_{n-1}^{(1,1)}}\right| < \dfrac{|c_{2,1}|}{r_1} < \dfrac{1}{4}$. Thus the inequality $|h_n^{(2,1)}| \geq \nu_2$ is valid. Moreover, $|R_n^{(2,1)}| \geq \nu_1\nu_2$.

By induction hypothesis the inequalities (19) hold for $q = s$, where $3 \leq s \leq N-1$. We write $R_n^{(q,1)}$ for $q = s+1$ as follows

$$R_n^{(s+1,1)} = R_n^{(s,1)} + \frac{c_{s+1,1}|}{|R_{n-1}^{(s,1)}} + \ldots + \frac{c_{s+1,1}|}{|R_0^{(s+1,1)}} = R_n^{(s,1)} h_n^{(s+1,1)}, \quad n \geq 1,$$

where $h_n^{(s+1,1)}$ is reversed continued fraction, that is defined by the formula (20). Its elements satisfy the conditions $\left|\dfrac{c_{s+1,1}}{R_n^{(s,1)} R_{n-1}^{(s,1)}}\right| < \dfrac{|c_{s+1,1}|}{\prod_{j=1}^q r_j} < \dfrac{1}{4}$, $r_j = \nu_j^2$. Thus, we have $|h_n^{(s+1,1)}| \geq \nu_{s+1}$, moreover, the following relations hold

$$\left|R_n^{(s+1,1)}\right| = \left|R_n^{(s,1)}\right| |h_n^{(s+1,1)}| > \prod_{j=1}^{s+1} \nu_j.$$

To prove the inequality (18) we have to estimate the following relations

$$\prod_{r=1}^{k_j} \frac{|c_{j,1}|}{|R_{l_j+2m-r}^{(j,1)} R_{l_j-r}^{(j,1)}|} = M_j \prod_{r=1}^{[k_j/2]} \frac{|c_{j,1}|}{|R_{l_j+2m-2r+1}^{(j,1)} R_{l_j+2m-2r}^{(j,1)}|} \cdot \prod_{r=1}^{[k_j/2]} \frac{|c_{j,1}|}{|R_{l_j-2r+1}^{(j,1)} R_{l_j-2r}^{(j,1)}|}, \quad j = \overline{2,N},$$

where k_j is defined by the formula (11). Since for the arbitrary natural n

$$\frac{|c_{j,1}|}{|R_n^{(j,1)}R_{n-1}^{(j,1)}|}=\frac{|c_{j,1}|/|R_n^{(j-1,1)}R_{n-1}^{(j-1,1)}|}{|h_n^{(j,1)}h_{n-1}^{(j,1)}|}<\frac{1/4}{v_j^2}=\frac{1}{(1+d_j)^2}=\rho_j,\quad j=\overline{2,N},$$

then for $n\geq 1$ and $m\geq 1$

$$|F_{n+2m}-F_n|\leq\sum_{k_1=0}^{n+1}\rho_1^{[k_1/2]}\rho^{n+1-k_1}\leq L\cdot\binom{N-1}{N+n-1}\cdot\frac{(\sqrt{\rho_1})^{n+1}-\rho^{n+1}}{\rho_1/\rho-\sqrt{\rho_1}}.$$

Finally, we obtain the truncation error bounds (18) for $m\to\infty$. □

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Боднар Д.І., Бубняк М.М. *Про збіжність* $(2,1,\ldots,1)$*-періодичного гіллястого ланцюгового дробу спеціального вигляду* // Карпатські матем. публ. — 2015. — Т.7, №2. — C. 148–154.

Означено $(2,1,\ldots,1)$-періодичний гіллястий ланцюговий дріб спеціального вигляду. Доведено ознаки збіжності 2-періодичного неперервного дробу та $(2,1,\ldots,1)$-періодичного дробу гіллястого ланцюгового дробу спеціального вигляду. Встановлено оцінку швидкості збіжності цього дробу при додаткових умовах.

Ключові слова і фрази: періодичні гіллясті ланцюгові дроби спеціального вигляду, збіжність.

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THE BARGMANN TYPE REDUCTION FOR SOME LAX INTEGRABLE TWO-DIMENSIONAL GENERALIZATION OF THE RELATIVISTIC TODA LATTICE

The possibility of applying the method of reducing upon finite-dimensional invariant subspaces, generated by the eigenvalues of the associated spectral problem, to some two-dimensional generalization of the relativistic Toda lattice with the triple matrix Lax type linearization is investigated. The Hamiltonian property and Lax-Liouville integrability of the vector fields, given by this system, on the invariant subspace related with the Bargmann type reduction are found out.

Key words and phrases: relativistic Toda lattice, triple Lax type linearization, invariant reduction, symplectic structure, Liouville integrability.

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INTRODUCTION

By use of the different Lie-algebraic approaches the Lax integrable $(2+1)$-dimensional nonlinear differential-difference systems given on functional manifolds of one discrete and one continuous independent variables have been obtained in [4], [10], [16], [26], [27]. The systems represented in the papers [10], [16], [26], [27] possess the triple Lax type linearizations and infinite sequences of local conservation laws. The $(2+1)$-dimensional nonlinear dynamical systems with such type properties on functional manifolds of two continuous independent variables have been investigated by means of the invariant reduction technique in the paper [14]. In this connection it is interesting to know whether the invariant reduction technique can be applied to the Lax integrable $(2+1)$-dimensional differential-difference systems obtained in [10], [16], [26], [27]. The reductions of the $(1+1)$-dimensional nonlinear differential-difference systems with the matrix Lax representations upon the finite-dimensional invariant subspaces generated by the critical points of the related local conservation laws and the associated spectral problem eigenvalues, have been considered in [13].

The aim of the present paper is to investigate the applicability of the invariant reduction technique to the $(2+1)$-dimensional differential-difference systems with the triple matrix Lax type linearizations, which can be obtained by means of two so called eigenfunction symmetries related with the same eigenvalue of the corresponding spectral problem (see [10]). This research is based on the approach to the study of the finite-dimensional invariant reductions for the $(1+1)$-dimensional nonlinear dynamical systems, possessing the matrix Lax type representations [6], [11], [23], and their superanalogs with the same properties, which has been devised in the papers [2], [8], [9], [11], [22], [21]. In the framework of such approach the exact

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symplectic structure on the invariant subspace can be found by means of the Gelfand-Dikii type relationship [5], [19] for the differential of the related Lagrangian function on a suitably extended phase space. The discrete analog of the Gelfand-Dikii relationship has been considered in [18], [19], [20].

In the present article the approach mentioned above is used to study the Bargmann type reduction [14] of the Lax integrable two-dimensional generalization of the relativistic Toda lattice [25], which has been constructed in [10].

The paper is organized in the following way. Section 1 contains the triple matrix Lax type linearization for this $(2+1)$-dimensional differential-difference system that will be used in further investigations. In section 2 we establish the existence of an exact symplectic structure on the Bargmann type invariant subspace by means of the discrete analog of the Gelfand-Dikii relationship as well as the Hamiltonian representations for the reduced commuting vector fields given by the system. In section 3, basing on the differential-geometric properties of the trace gradient for the monodromy matrix of the associated periodic matrix linear spectral problem, we obtain the corresponding Lax representations for the reduced vector fields. The complete set of the functionally independent conservation laws which are involutive with respect to the corresponding Poisson bracket and as a consequence ensure the Liouville integrability [1], [17] of the reduced vector fields is also found.

1 The triple matrix Lax type linearization for the two-dimensional generalization of the relativistic Toda lattice

In the paper [10] we have constructed the set of the hierarchies of the eigenfunction symmetries

$$dl/d\tau_{s,m} = -[M^s_m, l], \; df_j/d\tau_{s,m} = (-M^s_m + \delta^j_s l^s) f_j, \; df^*_j/d\tau_{s,m} = (M^s_m - \delta^j_s l^s)^* f^*_j, \tag{1}$$

which are additional homogeneous symmetries of the Lax type hierarchy on the extended dual space to the Lie algebra [3] of Laurent series by the usual shift operator $\mathcal{E}$

$$dl/dt_s = [l^s_+, l], \; df_j/dt_s = l^s_+ f_j, \; df^*_j/dt_s = -(l^s_+)^* f^*_j, \tag{2}$$

where $l := \mathcal{E} + \sum_{j=1}^R f_j \mathcal{E}(\mathcal{E}-1)^{-1} f^*_j$, $\mathbf{f} = (f_1, f_2, \dots, f_R, f^*_1, f^*_2, \dots, f^*_R)^\top \in \mathcal{M}^{2R}$,

$$\mathcal{M}^{2R} := \{\mathbf{g} : \mathbf{g}(n) \in \mathbb{C}^{2R}, \, \mathbf{g}(n+q) = \mathbf{g}(n), \, n \in \mathbb{Z}\}, \quad q \in \mathbb{N},$$

$$M^s_m := \sum_{\beta=0}^{s-1} (l^\beta f_m)(\mathcal{E}-1)^{-1}(l^{*(s-1-\beta)} f^*_m),$$

δ^j_m is the Kronecker symbol, $j, m = \overline{1, R}$, and the lower index "+" denotes a projection of the corresponding operator on the Lie subalgebra of power series, $t_m, \tau_{s,m} \in \mathbb{R}$, $s \in \mathbb{N}$. Here any operator $\mathcal{A}^*$ is assumed to be adjoint to the super-integro-differential one $\mathcal{A}$ with respect to the scalar product

$$(x, y) = \sum_{n \in \mathbb{Z}} y(n) \bar{z}(n),$$

where $y, z \in \ell_2(\mathbb{Z}; \mathbb{C})$, $n \in \mathbb{Z}$. In the paper the line over any variable denotes the complex conjugation of this variable.

In the case of $R = 1$ and $s = 2$ the evolutions of the functions describe the relativistic Toda lattice.

The vector fields (2) have been considered as the Hamiltonian flows generated by the Casimir functionals

$$\gamma_s = \frac{1}{s+1} \sum_{n=0}^{q-1} res \, l^{s+1}[\mathbf{f}(n)], \quad s \in \mathbb{Z}_+, \tag{3}$$

where the symbol "*res*" denotes the coefficient at $\mathcal{E}^0$ in the expansion of the corresponding operator, and Poisson structure found in [10]. In that paper the hierarchies (3) have been established to be Hamiltonian with respect to the natural powers of some different eigenvalues of the associated spectral problem and Poisson structure mentioned above. It has been shown also that for each $j = \overline{1, N}$ the first eigenfunction symmetry and any other which belong both to the hierarchy related with the same eigenvalue can be applied to construct (2+1)-dimensional differential-difference systems with the triple matrix Lax linearizations. These systems have been obtained by introducing some new functions which denote the expressions with inverse operator to the difference one into the equations of the eigenfunction symmetries.

In the present paper we consider two additional homogeneous symmetries for the Lax type hierarchy (2) such that

$$df_j/d\tau = (-M^1_1 + \delta^j_1 l) f_j, \; df^*_j/d\tau = (M^1_1 - \delta^j_1 l)^* f^*_j \tag{4}$$

and

$$df_j/dT = (l^2_+ - M^2_1 + \delta^j_1 l^2) f_j, \; df^*_j/dT = (-l^2_+ + M^2_1 - \delta^j_1 l^2)^* f^*_j, \tag{5}$$

where $\tau := \tau_{1,1}$ and $d/dT := d/dt_2 + \tau_{1,2}$, in the case of $R = 2$. The vector fields $d/d\tau$ and d/dT are commuting because of the relation

$$dl^2_+/d\tau = [l^2_+, M^1_1]_+ \tag{6}$$

where $l^2_+ := \mathcal{E}^2 + w_1 \mathcal{E} + w_0$, $w_1 := (\mathcal{E}P) + P$, $w_0 := P^2 + \sum_{j=1}^2 ((\mathcal{E} f_j) f^*_j + f_j(\mathcal{E}^{-1} f^*_j))$ and $P = \sum_{j=1}^2 f_j f^*_j$.

The dynamical systems (4), (5) and commutability condition (6) are written as

$$\begin{aligned} & f_{1,\tau} = (\mathcal{E} f_1) + P f_1 + u f_2, f^*_{1,\tau} = -(\mathcal{E}^{-1} f^*_1) - P f^*_1 + (\mathcal{E}\tilde{u}) f^*_2), \\ & f_{2,\tau} = -\tilde{u} f_1, \quad f^*_{2,\tau} = -(\mathcal{E} u) f^*_1, \end{aligned} \tag{7}$$

$$\begin{aligned} & f_{1,T} = f_{1,\tau\tau} + (\mathcal{E}^2 f_1) + w_1(\mathcal{E} f_1) + w_0 f_1 + 2(f_1(\mathcal{E}^{-1} f^*_1) + u\tilde{u}) f_1, \\ & f^*_{1,T} = -f^*_{1,\tau\tau} - (\mathcal{E}^{-2} f^*_1) - (\mathcal{E}^{-1} w_1)(\mathcal{E}^{-1} f^*_1) - w_0 f^*_1 - 2(f_1(\mathcal{E}^{-1} f^*_1) + u\tilde{u}) f^*_1, \\ & f_{2,T} = (\mathcal{E}^2 f_2) + w_1(\mathcal{E} f_2) + w_0 f_2 - \tilde{u} f_{1,\tau} + \tilde{u}_\tau f_1, \\ & f^*_{2,T} = -(\mathcal{E}^{-2} f^*_2) - (\mathcal{E}^{-1} w_1)(\mathcal{E}^{-1} f^*_2) - w_0 f^*_2 + u f^*_{1,\tau} - u_\tau f^*_1, \\ & (\mathcal{E} - 1) u = f_1 f^*_2, \quad (\mathcal{E} - 1)\tilde{u} = f^*_1 f_2, \end{aligned} \tag{8}$$

$$\begin{aligned} & w_{0,\tau} = (\mathcal{E}^2 f_1) f^*_1 - f_1(\mathcal{E}^{-2} f^*_1) + w_1(\mathcal{E} f_1) f^*_1 - f_1(\mathcal{E}^{-1} w_1)(\mathcal{E}^{-1} f^*_1), \\ & w_{1,\tau} = (\mathcal{E}^2 f_1)(\mathcal{E} f^*_1) - f_1(\mathcal{E}^{-1} f^*_1), \end{aligned} \tag{9}$$

where $u, \tilde{u}$ are some q-periodical complex-valued functions. The dynamical system (8) and relationships (9) describe the Lax integrable (2+1)-dimensional differential-difference system [10], which can be considered as some two-dimensional generalization of the relativistic Toda lattice.

Its triple Lax type linearization [10] is formed by the spectral relationship

$$ly = \lambda y, \tag{10}$$

where $y \in \ell_2(\mathbb{Z};\mathbb{C})$, $\lambda \in \mathbb{C}$ is a spectral parameter, and evolution equations

$$dy/d\tau = -M_1^1 y, \tag{11}$$
$$dy/dT = (l_+^2 - M_1^2)y. \tag{12}$$

The corresponding adjoint spectral relationship and adjoint evolutions take following forms:

$$l^* z = \lambda z, \tag{13}$$
$$dz/d\tau = M_1^{1*} z, \tag{14}$$
$$dz/dT = -(l_+^2 - M_1^2)^* z, \tag{15}$$

where $l^* = \mathcal{E}^{-1} - \sum_{j=1}^{2}(f_j^*(\mathcal{E}-1)^{-1} f_j)$. The spectral relationships (10) and (13) have the equivalent matrix forms

$$\mathcal{E}Y = AY, \tag{16}$$
$$\mathcal{E}^{-1}Z = (\mathcal{E}^{-1}A^\top)Z, \tag{17}$$

where $Y,\ Z \in \ell_2(\mathbb{Z};\mathbb{C}^3)$, $Y = (y_1, y_2, y_3)^\top$, $y_3 := y$, $Z = (z_1, z_2, z_3)^\top$, $z_3 := (\mathcal{E}^{-1}z)$, $A := A[\mathbf{f};\lambda]$ and

$$A = \begin{pmatrix} 1 & 0 & f_1^* \\ 0 & 1 & f_2^* \\ -f_1 & -f_2 & \lambda - P \end{pmatrix}.$$

The corresponding evolutions are written as

$$dY/d\tau = B^{(\tau)}Y, \quad dZ/d\tau = -(B^{(\tau)})^\top Z, \tag{18}$$
$$dY/dT = B^{(T)}Y, \quad dZ/dT = -(B^{(T)})^\top Z, \tag{19}$$

where $B^{(\tau)} := B^{(\tau)}[\mathbf{f};\lambda]$, $B^{(T)} := B^{(T)}[\mathbf{f};\lambda]$, and

$$B^{(\tau)} = \begin{pmatrix} -\lambda & \tilde{u} & (\mathcal{E}^{-1}f_1^*) \\ -u & 0 & 0 \\ -f_1 & 0 & 0 \end{pmatrix},$$

$$B^{(T)} = \begin{pmatrix} \begin{array}{l} -\lambda^2 - u\tilde{u} - \\ -2f_1(\mathcal{E}^{-1}f_1^*) \end{array} & \begin{array}{l} \lambda\tilde{u} - \tilde{u}_\tau - \\ -f_2(\mathcal{E}^{-1}f_1^*) \end{array} & \begin{array}{l} 2\lambda(\mathcal{E}^{-1}f_1^*) - \tilde{u}(\mathcal{E}^{-1}f_2^*) + \\ +2(\mathcal{E}^{-1}P)(\mathcal{E}^{-1}f_1^*) + \\ +2(\mathcal{E}^{-2}f_1^*) \end{array} \\ \begin{array}{l} -\lambda u - u_\tau - \\ -f_1(\mathcal{E}^{-1}f_2^*) \end{array} & \begin{array}{l} u\tilde{u} - \\ -f_2(\mathcal{E}^{-1}f_2^*) \end{array} & \begin{array}{l} \lambda(\mathcal{E}^{-1}f_2^*) + u(\mathcal{E}^{-1}f_1^*) + \\ +(\mathcal{E}^{-2}f_2^*) + \\ +(\mathcal{E}^{-1}P)(\mathcal{E}^{-1}f_2^*) \end{array} \\ \begin{array}{l} -2\lambda f_1 - uf_2 \\ -2(\mathcal{E}f_1) - 2Pf_1 \end{array} & \begin{array}{l} -\lambda f_2 - \tilde{u}f_1 - \\ -(\mathcal{E}f_2) - Pf_2 \end{array} & \begin{array}{l} \lambda^2 + 2f_1(\mathcal{E}^{-1}f_1^*) + \\ +f_2(\mathcal{E}^{-1}f_2^*) \end{array} \end{pmatrix}.$$

The matrices $B^{(\tau)}$ and $B^{(T)}$ satisfy the compatibility conditions

$$dA/d\tau = (\mathcal{E}B^{(\tau)})A - AB^{(\tau)}, \tag{20}$$
$$dA/dT = (\mathcal{E}B^{(T)})A - AB^{(T)}. \tag{21}$$

The system (8)-(9) possesses the infinite sequence of the local conservation laws (3).

2 THE SYMPLECTIC STRUCTURE ON SOME INVARIANT SUBSPACE

We will study below the differential-geometric properties of the commuting vector fields $d/d\tau$ and d/dT on their common invariant finite-dimensional subspace $\mathcal{M}_N^4 \subset \mathcal{M}^4$ such as

$$\mathcal{M}_N^4 = \left\{\mathbf{f} \in \mathcal{M}^4 :\ grad\,\mathcal{L}_N[\mathbf{f}(n)] = 0\right\},\ \mathcal{L}_N := \sum_{n=0}^{q-1} \mathcal{L}_N[\mathbf{f}(n)] = -\gamma_0 + \sum_{i=1}^{N} c_i\lambda_i,$$

where $\gamma_0 = \sum_{n=0}^{q-1}\sum_{j=1}^{2} f_j(n)f_j^*(n)$, $\lambda_i \in \mathbb{C}$, $i = \overline{1,N}$, are different eigenvalues of the periodic spectral problem (16) with the corresponding eigenvectors $Y_i = (y_{1i}, y_{2i}, y_{3i})^\top \in \mathcal{W}$ and adjoint eigenvectors $Z_i = (z_{1i}, z_{2i}, z_{3i})^\top \in \mathcal{W}$, $\mathcal{W} := \{\mathbf{a} = (a_1, a_2, a_3)^\top :\ \mathbf{a}(n) \in \mathbb{C}^3,\ \mathbf{a}(n+q) = \mathbf{a}(n+q),\ n \in \mathbb{Z}\} \subset \ell_2(\mathbb{Z};\mathbb{C})$, and $c_i \in \mathbb{C} \setminus \{0\}$, $i = \overline{1,N}$, are some fixed constants, which will be chosen later. Here the eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1,N}$, are considered as smooth by Frechet functionals on $\mathcal{M}^4$.

We will first analyze the differential-geometric structure of the invariant subspace $\mathcal{M}_N^4 \subset \mathcal{M}^4$. To describe this subspace explicitly we will find the gradients of the eigenvalues $\lambda_i \in \mathcal{D}(\mathcal{M}^4)$, $i = \overline{1,N}$.

Because of the relations

$$\sum_{n=0}^{q-1}(\mathcal{E}Y_i(n))^\top(\mathcal{E}Z_i(n)) = \sum_{n=0}^{q-1}(A[\mathbf{f}(n);\lambda_i]Y_i(n))^\top(\mathcal{E}Z_i(n)), \quad i = \overline{1,N}, \tag{22}$$

that follow from the spectral problem (16), we can derive the explicit form of the gradient of the eigenvalue λ_i for any $i = \overline{1,N}$ only on the level surface $\{(\mathbf{f}, \mathcal{Y}, \mathcal{Z})^\top \in \hat{\mathcal{M}}^4 :\ \mu_i = a_i,\ a_i \in \mathbb{C} \setminus \{0\}\}$ of the functional $\mu_i := -\sum_{n=0}^{q-1} y_{3i}(n)(\mathcal{E}z_{3i}(n))$, which is invariant with respect to the vector fields $d/d\tau$ and d/dT. Thus, for any $i = \overline{1,N}$ the gradient of the eigenvalue λ_i on this level surface is written as

$$grad\,\lambda_i = \begin{pmatrix} \delta\lambda_i/\delta f_1 \\ \delta\lambda_i/\delta f_2 \\ \delta\lambda_i/\delta f_1^* \\ \delta\lambda_i/\delta f_2^* \end{pmatrix} = -\frac{1}{a_i}\begin{pmatrix} \bar{f}_1^*\bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) + \bar{y}_{1i}(\mathcal{E}\bar{z}_{3i}) \\ \bar{f}_2^*\bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) + \bar{y}_{2i}(\mathcal{E}\bar{z}_{3i}) \\ \bar{f}_1\bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) - \bar{y}_{3i}(\mathcal{E}\bar{z}_{1i}) \\ \bar{f}_2\bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) - \bar{y}_{3i}(\mathcal{E}\bar{z}_{2i}) \end{pmatrix},$$

where $\bar{Y}_i = (\bar{y}_{1i}, \bar{y}_{2i}, \bar{y}_{3i})^\top$, $\bar{Z}_i = (\bar{z}_{1i}, \bar{z}_{2i}, \bar{z}_{3i})^\top$, $i = \overline{1,N}$.

Let us choose $a_i = -c_i$, $i = \overline{1,N}$, and investigate the vector fields $d/d\tau$ and d/dT on the invariant finite-dimensional subspace $\mathcal{M}_N^4 \bigcap H_c \subset \mathcal{M}^4$ given by the following Bargmann type constraints

$$\mathcal{M}_N^4 \bigcap H_c = \left\{ \mathbf{f} \in \mathcal{M}^4 :\ \rho f_1 = -\sum_{i=1}^{N} y_{3i}\bar{z}_{1i},\ \rho f_2 = -\sum_{i=1}^{N} y_{3i}\bar{z}_{2i}, \right.$$
$$\left. \rho f_1^* = \sum_{i=1}^{N} y_{1i}\bar{z}_{3i},\ \rho f_2^* = \sum_{i=1}^{N} y_{2i}\bar{z}_{3i} \right\},$$

where $H_c := \{(\mathbf{f}, \mathcal{Y}, \mathcal{Z})^\top \in \hat{\mathcal{M}}^4 : \ \mu_i = -c_i, \ c_i \in \mathbb{C} \setminus \{0\}, \ i = \overline{1,N}\}$ is a common level surface of the invariant functionals μ_i, $i = \overline{1,N}$, in the extended phase space $\hat{\mathcal{M}}^4 := \mathcal{M}^4 \times W^{2N}$ of the coupled dynamical systems (8), (9), (18) and (19) with the parameter $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, and $\mathcal{Y} := (Y_1, Y_2, \ldots, Y_N)^\top$, $\mathcal{Z} := (Z_1, Z_2, \ldots, Z_N)^\top$, $\tilde{Z}_i := \mathcal{E} Z_i = (\tilde{z}_{1i}, \tilde{z}_{2i}, \tilde{z}_{3i})^\top$, $i = \overline{1,N}$, $\rho = 1 - \sum_{i=1}^N y_{3i}\tilde{z}_{3i}$. This invariant subspace can be described by means of the equivalent relationships

$$\mathcal{M}_N^4 \bigcap H_c = \Big\{ \mathbf{f} \in \mathcal{M}^4 : \ f_1 = -\sum_{i=1}^N y_{3i} z_{1i}, \ f_2 = -\sum_{i=1}^N y_{3i} z_{2i}, \quad \mathcal{E}^{-1} f_1^* = \sum_{i=1}^N y_{1i} z_{3i}, \ \mathcal{E}^{-1} f_2^* = \sum_{i=1}^N y_{2i} z_{3i} \Big\}, \tag{23}$$

From (23) it follows that the functions f_1, f_2, $\mathcal{E}^{-1}f_1^*$, $\mathcal{E}^{-1}f_2^*$ are expressed via the coordinates of the eigenvectors Y_i and Z_i, $i = \overline{1,N}$, on the invariant subspace $\mathcal{M}_N^4 \bigcap H_c$. The relation

$$\begin{aligned}
&\sum_{i=1}^N y_{1i} z_{2i} = 0, \quad \sum_{i=1}^N y_{2i} z_{1i} = 0, \quad (\mathcal{E} - 1) \sum_{i=1}^N y_{3i} z_{3i} = 0, \\
&u \sum_{i=1}^N (y_{2i} z_{2i} - y_{1i} z_{1i}) = -\sum_{i=1}^N \lambda_i y_{2i} z_{1i} - f_1(\mathcal{E}^{-1} f_2^*), \\
&\tilde{u} \sum_{i=1}^N (y_{2i} z_{2i} - y_{1i} z_{1i}) = \sum_{i=1}^N \lambda_i y_{1i} z_{2i} + f_2(\mathcal{E}^{-1} f_1^*), \\
&u_\tau \sum_{i=1}^N (y_{1i} z_{1i} - y_{2i} z_{2i}) = \sum_{i=1}^N \lambda_i^2 y_{2i} z_{1i} + u \sum_{i=1}^N (\lambda_i y_{2i} z_{2i} - \lambda_i y_{1i} z_{1i}) \\
&\quad + u(-f_1(\mathcal{E}^{-1} f_1^*) + f_2(\mathcal{E}^{-1} f_2^*)) + (\mathcal{E}^{-1} f_2^*) \sum_{i=1}^N \lambda_i y_{3i} z_{1i} + 2 f_1 \sum_{i=1}^N \lambda_i y_{2i} z_{3i} \\
&\quad + f_1(\mathcal{E}^{-1} f_2^*) \sum_{i=1}^N (y_{1i} z_{1i} - y_{2i} z_{2i}) - (\mathcal{E}^{-1} P)(\mathcal{E}^{-1} f_2^*) f_1, \\
&\tilde{u}_\tau \sum_{i=1}^N (y_{2i} z_{2i} - y_{1i} z_{1i}) = \sum_{i=1}^N \lambda_i^2 y_{1i} z_{2i} + \tilde{u} \sum_{i=1}^N (\lambda_i y_{1i} z_{1i} - \lambda_i y_{2i} z_{2i}) \\
&\quad - \tilde{u}(-f_1(\mathcal{E}^{-1} f_1^*) + f_2(\mathcal{E}^{-1} f_2^*)) + (\mathcal{E}^{-1} f_1^*) \sum_{i=1}^N \lambda_i y_{3i} z_{2i} - f_2 \sum_{i=1}^N \lambda_i y_{1i} z_{3i} \\
&\quad - f_2(\mathcal{E}^{-1} f_1^*) \sum_{i=1}^N (y_{1i} z_{1i} + y_{2i} z_{2i} - 2 y_{3i} z_{3i}) - 2(\mathcal{E}^{-1} P)(\mathcal{E}^{-1} f_1^*) f_2, \\
&\mathcal{E} f_1 = -\sum_{i=1}^N \lambda_i y_{3i} z_{1i} + f_1 \sum_{i=1}^N (y_{1i} z_{1i} - y_{3i} z_{3i}) - P f_1, \\
&\mathcal{E} f_2 = -\sum_{i=1}^N \lambda_i y_{3i} z_{2i} + f_2 \sum_{i=1}^N (y_{2i} z_{2i} - y_{3i} z_{3i}) - P f_2, \\
&\mathcal{E}^{-2} f_1^* = \sum_{i=1}^N \lambda_i y_{1i} z_{3i} + (\mathcal{E}^{-1} f_1^*) \Big(\sum_{i=1}^N (y_{1i} z_{1i} - y_{3i} z_{3i}) - (\mathcal{E}^{-1} P) \Big), \\
&\mathcal{E}^{-2} f_2^* = \sum_{i=1}^N \lambda_i y_{2i} z_{3i} + (\mathcal{E}^{-1} f_2^*) \Big(\sum_{i=1}^N (y_{2i} z_{2i} - y_{3i} z_{3i}) - (\mathcal{E}^{-1} P) \Big),
\end{aligned} \tag{24}$$

obtained with taking into account the equations (8), (9), spectral problems (16), (17) and evolutions (23), allow to express the entries of the matrices $B^\tau[\mathbf{f};\lambda]$ and $B^T[\mathbf{f};\lambda]$, reduced upon $\mathcal{M}_N^4 \bigcap H_c$, via the coordinates of the eigenvectors Y_i and Z_i, $i = \overline{1,N}$. In addition, from the spectral problems (16), (17) and evolution equations (23), when $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ we have

$$\frac{d}{d\tau} \sum_{\chi=1}^3 y_{\chi i} z_{\chi i} = 0, \quad \frac{d}{dT} \sum_{\chi=1}^3 y_{\chi i} z_{\chi i} = 0, \quad \frac{d}{d\tau} \sum_{i=1}^N y_{3i} \tilde{z}_{3i} = 0, \quad \frac{d}{dT} \sum_{i=1}^N y_{3i} \tilde{z}_{3i} = 0.$$

Therefore, we are in a position to formulate the following theorem.

Theorem 1. *The commuting vector fields $d/d\tau$ and d/dT, given by the system (8)-(9), allow the invariant reductions upon the finite-dimensional subspaces $\mathcal{M}_N^4 \bigcap H_c \subset \mathcal{M}^4$, $N \in \mathbb{N}$. These subspaces are diffeomorphic to the finite-dimensional space $\mathcal{M}_\mathcal{F}$, which is smoothly embedded into the space $\mathbb{R}^{6N}$ and endowed with the Poisson bracket $\{.,.\}_{\omega_\mathcal{F}^{(2)}}$, being the Dirac reduction of the Poisson bracket $\{.,.\}_{\hat{\omega}^{(2)}}$ related with the symplectic structure*

$$\hat{\omega}^{(2)} = \sum_{i=1}^N \sum_{s=1}^3 d(\mathcal{E}^{-1} \tilde{z}_{si}) \wedge dy_{si} = \sum_{i=1}^N \sum_{s=1}^3 dz_{si} \wedge dy_{si}, \tag{25}$$

where "$\wedge$" is a symbol of the exterior product on the Grassmann algebra of differential forms on $\mathbb{C}^{6N}$. The reduced vector fields $d/d\tau$ and d/dT, given by the equations (18) and (19) when $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, are Hamiltonian with respect to the Poisson bracket $\{.,.\}_{\omega_\mathcal{F}^{(2)}}$. The corresponding Hamiltonians $h^{(\tau)}$, $h^{(T)} \in C^\infty(\mathbb{R}^{6N};\mathbb{R})$ are reductions of the functions $\hat{h}^{(\tau)}$, $\hat{h}^{(T)} \in D(\hat{\mathcal{M}}^4)$, satisfying the equalities

$$\Big\langle (d\mathbf{f}/d\tau, d\mathcal{Y}/d\tau, d\tilde{\mathcal{Z}}/d\tau)^\top, \ grad\, \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] \Big\rangle = -(\mathcal{E} - 1)\hat{h}^{(\tau)}, \tag{26}$$

$$\Big\langle (d\mathbf{f}/dT, d\mathcal{Y}/dT, d\tilde{\mathcal{Z}}/dT)^\top, \ grad\, \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] \Big\rangle = -(\mathcal{E} - 1)\hat{h}^{(T)}, \tag{27}$$

where the brackets $\langle , \rangle$ denote the standard scalar product on $\mathbb{C}^{6N+4}$, and involutive with respect to the Poisson bracket $\{.,.\}_{\omega^{(2)}}$. The relationships (23) describe all periodic and quasi-periodic solutions of the system (8), (9) on the subspaces $\mathcal{M}_N^4 \bigcap H_c$, $N \in \mathbb{N}$.

Proof. The exact symplectic structure on the invariant subspace $\mathcal{M}_N^4 \subset \mathcal{M}^4$ can be found by means of the discrete analog [18], [19], [20] of the Gelfand-Dikii relationship on the functional manifold $\mathcal{M}^4$ in the same manner as has been done in the paper [19] for the subspaces of critical points of local conservation laws.

To make use this relationship we need the explicit forms of the smooth by Frechet functionals λ_i, $i = \overline{1,N}$, on H_c. From the equalities (22) we have

$$\begin{aligned}
\lambda_i' = \sum_{n=0}^{q-1} \Big(&\sum_{s=1}^3 (\mathcal{E} y_{si}(n)) \tilde{z}_{si}(n) - y_{1i}(n) \tilde{z}_{1i}(n) - y_{2i}(n) \tilde{z}_{2i}(n) - f_1^*(n) y_{3i}(n) \tilde{z}_{1i}(n) \\
&- f_2^*(n) y_{3i}(n) \tilde{z}_{2i}(n) + f_1(n) y_{1i}(n) \tilde{z}_{3i}(n) + f_2(n) y_{2i}(n) \tilde{z}_{3i}(n) + P(n) y_{3i}(n) \tilde{z}_{3i}(n) \Big),
\end{aligned}$$

where $\lambda_i' := \lambda_i|_{H_c}$, $i = \overline{1,N}$, on the level surface H_c in the extended phase space $\hat{\mathcal{M}}^4$. Since the functionals $\lambda_i' \in \mathcal{D}(\hat{\mathcal{M}}^4)$, $i = \overline{1,N}$, depend on the functions $(f, \mathcal{Y}, \mathcal{Z})^\top \in \hat{\mathcal{M}}^4$, it is expedient to apply the discrete analog of the Gelfand-Dikii relationship to the Lagrangian functional $\hat{\mathcal{L}}_N := \sum_{n=0}^{q-1} \hat{\mathcal{L}}_N[\mathbf{f}(n), \mathcal{Y}(n), \bar{\mathcal{Z}}(n)] \in \mathcal{D}(\hat{\mathcal{M}}^4)$ of the form

$$\hat{\mathcal{L}}_N = -\gamma_0 + \sum_{i=1}^{N} \lambda_i' + \sum_{i=1}^{N} \varsigma_i \mu_i,$$

where $\varsigma_i \in \mathbb{C}$ are Lagrangian multipliers, $\mu_i = -\sum_{n=0}^{q-1} y_{3i}(n)\bar{z}_{3i}(n)$, $i = \overline{1,N}$.

Because of the Lax theorem [11], [12] the condition $grad\, \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \bar{\mathcal{Z}}] = 0$ determines the invariant subspace $\tilde{\mathcal{M}}_N^4 \subset \hat{\mathcal{M}}^4$,

$$\tilde{\mathcal{M}}_N^4 = \Big\{ (\mathbf{f}, \mathcal{Y}, \bar{\mathcal{Z}})^\top \in \mathcal{M}^4 : \ f_1 = -\sum_{i=1}^{N} y_{3i} z_{1i}, \ f_2 = -\sum_{i=1}^{N} y_{3i} z_{2i}, \ \mathcal{E}^{-1} f_1^* = \sum_{i=1}^{N} y_{1i} z_{3i},$$
$$\mathcal{E}^{-1} f_2^* = \sum_{i=1}^{N} y_{2i} z_{3i}, \mathcal{E} Y_i = A[\mathbf{f}; \varsigma_i] Y_i, \ \mathcal{E}^{-1} \bar{Z}_i = A^\top[\mathbf{f}; \varsigma_i] \bar{Z}_i, \ i = \overline{1,N} \Big\},$$

of the coupled dynamical systems (8), (9), (18) and (19) with the parameter $\lambda \in \{\varsigma_1, \ldots, \varsigma_N\}$. Thus, for every $N \in \mathbb{N}$ the invariant subspace $\mathcal{M}_N^4 \cap H_c \subset \mathcal{M}^4$ is diffeomorphic to the subspace $\tilde{\mathcal{M}}_N^4 \subset \hat{\mathcal{M}}^4$ when $\varsigma_i = \lambda_i$, $i = \overline{1,N}$.

By means of the discrete analog of the Gelfand-Dikii differential relationship [18], [19], [20] for $\hat{\mathcal{L}}_N \in \mathcal{D}(\hat{\mathcal{M}}^4)$ such as

$$d\hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \bar{\mathcal{Z}}] = \langle (d\mathbf{f}, d\mathcal{Y}, d\bar{\mathcal{Z}})^\top, grad\, \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \bar{\mathcal{Z}}] \rangle + ((\mathcal{E} - 1)\alpha^{(1)}), \tag{28}$$

where $(\mathcal{Y}, \bar{\mathcal{Z}})^\top$ are coordinates on the suitably truncated manifold $\hat{\mathcal{M}}_N^4$ and the brackets $\langle \,, \rangle$ denote the standard scalar product on $\mathbb{C}^{6N+4}$, we can find the exact two-form (25)

$$\hat{\omega}^{(2)} := d\alpha.^{(1)}$$

The reduced two-form $\omega^{(2)} := \hat{\omega}^{(2)}\big|_{\tilde{\mathcal{M}}_N^4}$ defines the symplectic structure on the invariant subspace $\mathcal{M}_N^4 \cap H_c \simeq \tilde{\mathcal{M}}_N^4 \subset \hat{\mathcal{M}}_N^4$, which is smoothly embedded into $\hat{\mathcal{M}}_N^4$ due to the relationships (23).

The formula (28) ensures the invariance of the reduced two-form $\omega^{(2)}$ with respect to the operator $(\mathcal{E} - 1)$, that is

$$\sum_{i=1}^{N}\sum_{s=1}^{3} d(\mathcal{E} z_{si}) \wedge d(\mathcal{E} y_{si}) = \sum_{i=1}^{N}\sum_{s=1}^{3} dz_{si} \wedge dy_{si}.$$

Taking into account that the subspace $\mathcal{M}_N^4 \cap H_c \subset \mathcal{M}^4$ is diffeomorphic to the finite-dimensional submanifold $\mathcal{M}_\mathcal{F} \subset \mathbb{R}^{6N}$ determined by the constraints

$$F_1 := \sum_{i=1}^{N} y_{1i} z_{2i} = 0, \quad F_2 := \sum_{i=1}^{N} y_{2i} z_{1i} = 0,$$

in the space $\mathbb{R}^{6N}$, we can obtain the symplectic structure on $\mathcal{M}_N^4 \cap H_c$ as a natural Dirac type reduction of the two-form $\hat{\omega}^{(2)}$ on $\mathcal{M}_\mathcal{F}$.

The two-form $\hat{\omega}^{(2)}$ generates the standard Poisson bracket $\{.,.\}_{\hat{\omega}^{(2)}}$ on $\mathbb{R}^{6N}$. As the matrix of constraints $\{F_{\kappa_1}, F_{\kappa_2}\}_{\hat{\omega}^{(2)}}$, $\kappa_1, \kappa_2 = 1,2$, is nondegenerate when $Q := \sum_{i=1}^{N}(y_{1i}z_{1i} - y_{2i}z_{2i}) \neq 0$,

the standard Dirac type reduction procedure [7,11] entails the Poisson bracket related with the symplectic structure $\omega_\mathcal{F}^{(2)} := \omega^{(2)}$ such that

$$\begin{aligned}
\{F, G\}_{\omega_\mathcal{F}^{(2)}} &= \{F, G\}_{\hat{\omega}^{(2)}} + \frac{1}{Q}\{F, F_1\}_{\hat{\omega}^{(2)}}\{F_2, G\}_{\hat{\omega}^{(2)}} - \frac{1}{Q}\{F, F_2\}_{\hat{\omega}^{(2)}}\{F_1, G\}_{\hat{\omega}^{(2)}} \\
&= \{F, G\}_{\hat{\omega}^{(2)}} + \frac{1}{Q}\sum_{i_1=1}^{N}\left(z_{2i_1}\frac{\partial F}{\partial z_{1i_1}} - y_{1i_1}\frac{\partial F}{\partial y_{2i_1}}\right)\sum_{i_2=1}^{N}\left(y_{2i_2}\frac{\partial G}{\partial y_{1i_2}} - z_{1i_2}\frac{\partial G}{\partial z_{2i_2}}\right) \\
&- \frac{1}{Q}\sum_{i_1=1}^{N}\left(-y_{2i_1}\frac{\partial F}{\partial y_{1i_1}} + z_{1i_1}\frac{\partial F}{\partial z_{2i_1}}\right)\sum_{i_2=1}^{N}\left(-z_{2i_2}\frac{\partial G}{\partial z_{1i_2}} + y_{1i_2}\frac{\partial_l G}{\partial y_{2i_2}}\right),
\end{aligned}$$

where $F, G \in C^\infty(\mathbb{R}^{6N}; \mathbb{R})$ are arbitrary smooth functions. Since

$$d\hat{\mathcal{L}}_N/d\tau = 0, \quad d\hat{\mathcal{L}}_N/dT = 0,$$

with taking into account the results obtained in the papers [18], [19] we can state the existence of the smooth by Frechet functions $\hat{h}^{(\tau)}, \hat{h}^{(T)} \in \mathcal{D}(\mathcal{M}^4)$, which satisfy the relations (26) and (27) correspondingly. Then for the functions $h^{(\tau)} := \hat{h}^{(\tau)}\big|_{\mathcal{M}_N^4}$ and $h^{(T)} := \hat{h}^{(T)}\big|_{\mathcal{M}_N^4}$ we have

$$i_{d/d\tau}\omega^{(2)} = -dh^{(\tau)}, \quad i_{d/dT}\omega^{(2)} = -dh^{(T)},$$

where $i_{d/d\tau}$ and $i_{d/dT}$ are inner differentiations with respect to the vector fields $d/d\tau : \mathcal{M}_N^4 \to T(\mathcal{M}_N^4)$ and $d/dT : \mathcal{M}_N^4 \to T(\mathcal{M}_N^4)$ in the Grassmann algebra of differential forms on $\mathbb{R}^{6N}$.

Therefore, the functions $h^{(\tau)}$ and $h^{(T)}$ are Hamiltonians of the reduced upon $\mathcal{M}_N^4 \cap H_c \subset \mathcal{M}^4$ vector fields $d/d\tau$ and d/dT when $\varsigma_i = \lambda_i$, $i = \overline{1,N}$. They take the following forms

$$\begin{aligned}
h^{(\tau)} &= -\sum_{i=1}^{N}\lambda_i y_{1i} z_{1i} - f_1(\mathcal{E}^{-1} f_1^*), \\
h^{(T)} &= \sum_{i=1}^{N}(\lambda_i^2 y_{3i} z_{3i} - \lambda_i^2 y_{1i} z_{1i}) \\
&+ \frac{\left(\sum_{i=1}^{N}\lambda_i y_{1i} z_{2i} + f_2(\mathcal{E}^{-1} f_1^*)\right)\left(\sum_{i=1}^{N}\lambda_i y_{2i} z_{1i} - f_1(\mathcal{E}^{-1} f_2^*)\right)}{\sum_{i=1}^{N}(y_{2i} z_{2i} - y_{1i} z_{1i})} \\
&+ 2(\mathcal{E}^{-1} f_1^*)\sum_{i=1}^{N}\lambda_i y_{3i} z_{1i} + (\mathcal{E}^{-1} f_2^*)\sum_{i=1}^{N}\lambda_i y_{3i} z_{2i} - 2f_1\sum_{i=1}^{N}\lambda_i y_{1i} z_{3i} - f_2\sum_{i=1}^{N}\lambda_i y_{2i} z_{3i} \\
&+ 2f_1(\mathcal{E}^{-1} f_1^*)\sum_{i=1}^{N}(y_{3i} z_{3i} - y_{1i} z_{1i}) + f_2(\mathcal{E}^{-1} f_2^*)\sum_{i=1}^{N}(y_{3i} z_{3i} - y_{1i} z_{1i}),
\end{aligned}$$

where the functions f_1, f_2, $\mathcal{E}^{-1} f_1^*$, $\mathcal{E}^{-1} f_2^*$ have the forms (23).

By means of the direct calculations it is easily to verify that

$$\{h^{(\tau)}, h^{(T)}\}_{\omega^{(2)}} = -\frac{d}{d\tau} h^{(T)} = 0.$$

□

Let us consider the vector field d/dt_1, commuting with the vector fields $d/d\tau$ and d/dT, on the functional manifold $\mathcal{M}^4$ and investigate its reduction upon the invariant subspace $\mathcal{M}^4_N \bigcap H_c \subset \mathcal{M}^4$, $N \in \mathbb{N}$. In the same manner as in the proof of Theorem 1 we can find the Hamiltonian representation for the reduced vector field d/dt_1. The corresponding Hamiltonian $h^{(t_1)}$ takes the form

$$h^{(t_1)} = \sum_{i=1}^{N} \lambda_i y_{3i} z_{3i} - f_1(\mathcal{E}^{-1} f_1^*) - f_2(\mathcal{E}^{-1} f_2^*).$$

Since

$$\{h^{(t_1)}, h^{(\tau)}\}_{\omega^{(2)}} = -\frac{d}{dt_1} h^{(\tau)} = 0, \quad \{h^{(t_1)}, h^{(T)}\}_{\omega^{(2)}} = -\frac{d}{dt_1} h^{(T)} = 0,$$

the reduced vector fields d/dt_1, $d/d\tau$ and d/dT are integrable in the case of $N = 1$ due to the Liouville theorem [1], [17].

3 THE LAX-LIOUVILLE INTEGRABILITY OF REDUCED VECTOR FIELDS

To state the Liouville integrability of the Hamiltonian vector fields $d/d\tau$ and d/dT on $\mathcal{M}^4_N \bigcap H_c \subset \mathcal{M}^4$ for all $N \in \mathbb{N}$ we need to construct the related matrix Lax representations, which depend on the spectral parameter $\lambda \in \mathbb{C}$, making use the reduction procedure for the monodromy matrix of the periodic spectral problem (16). Thus, the following theorem holds.

Theorem 2. *For every $N \in \mathbb{N}$ on the intersections of the finite-dimensional subspace $\mathcal{M}^4_N \bigcap H_c \simeq \mathcal{M}_{\mathcal{F}}$ with the level surfaces $h_C := \{(\mathcal{Y}, \mathcal{Z})^\top \in \mathbb{R}^{6N} : \sum_{i=1}^N y_{3i}\bar{z}_{31} = C, C \in \mathbb{C}\}$ of the invariant function $1 - \rho = \sum_{i=1}^N y_{3i}\bar{z}_{3i}$ the matrix Lax representations for the Hamiltonian vector fields $d/d\tau$ and d/dT have the following forms*

$$d\check{S}_N/d\tau = [B_N^{(\tau)}, \check{S}_N], \tag{29}$$

$$d\check{S}_N/dT = [B_N^{(T)}, \check{S}_N], \tag{30}$$

where $B_N^{(\tau)} := B_N^{(\tau)}(\mathcal{Y}, \mathcal{Z}; \lambda) = B^{(\tau)}[\mathbf{f}; \lambda]\Big|_{\mathcal{M}_{\mathcal{F}} \bigcap h_C}$, $B_N^{(T)} := B_N^{(T)}(\mathcal{Y}, \mathcal{Z}; \lambda) = B^{(T)}[\mathbf{f}; \lambda]\Big|_{\mathcal{M}_{\mathcal{F}} \bigcap h_C}$ are projections of the corresponding matrices on $\mathcal{M}_{\mathcal{F}} \bigcap h_C$ and

$$\check{S}_N = \sum_{i=1}^{N} \frac{S_i}{\lambda - \lambda_i} + S_0 = \sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i} \begin{pmatrix} y_{1i}z_{1i} & y_{1i}z_{2i} & y_{1i}z_{3i} \\ y_{2i}z_{1i} & y_{2i}z_{2i} & y_{2i}z_{3i} \\ y_{3i}z_{1i} & y_{3i}z_{2i} & y_{3i}z_{3i} \end{pmatrix} + \begin{pmatrix} -C & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & 1-C \end{pmatrix}. \tag{31}$$

Proof. Making use the spectral problem (16), we can express the gradient $\varphi(n; \bar{\lambda}) := grad \operatorname{tr} S$ of the trace of the corresponding monodromy matrix

$$S := S(n; \lambda) = A[\mathbf{f}(n+q-1); \lambda] A[\mathbf{f}(n+q-2); \lambda] \times \ldots \times A[\mathbf{f}(n); \lambda]$$

via the entries of the matrix $V = SA^{-1}$ by such a way

$$\varphi(n; \bar{\lambda}) = \begin{pmatrix} \operatorname{tr}(\bar{V}\bar{A}_{f_1}) \\ \operatorname{tr}(\bar{V}\bar{A}_{f_2}) \\ \operatorname{tr}(\bar{V}\bar{A}_{f_1^*}) \\ \operatorname{tr}(\bar{V}\bar{A}_{f_2^*}) \end{pmatrix} = \begin{pmatrix} -\bar{V}_{13} - \bar{f}_1^* \bar{V}_{33} \\ -\bar{V}_{23} - \bar{f}_2^* \bar{V}_{33} \\ \bar{V}_{31} - \bar{f}_1 \bar{V}_{33} \\ \bar{V}_{32} - \bar{f}_2 \bar{V}_{33} \end{pmatrix},$$

where $\varphi(n; \bar{\lambda}) \simeq \sum_{r \in \mathbb{Z}_+} \varphi_r(n) \bar{\lambda}^{-(r+1)}$, $\varphi_r = grad\, \gamma_r[\mathbf{f}]$, when $|\lambda| \to \infty$, $\bar{V}$ is a matrix with the entries, being complex conjugate to the corresponding ones of the matrix

$$V := \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

and $\bar{A}_{f_1}$, $\bar{A}_{f_2}$, $\bar{A}_{f_1^*}$, $\bar{A}_{f_2^*}$ are matrices with the enteries, being complex conjugate to the corresponding ones of A_{f_1}, A_{f_2}, $A_{f_1^*}$, $A_{f_2^*}$ respectively.

From the equation for the matrix V

$$\mathcal{E}(VA) = AV, \tag{32}$$

we can obtain the Magri type relationships [15]

$$\vartheta\, \varphi(n; \bar{\lambda}) = \bar{\lambda}\, \eta\, \varphi(n; \bar{\lambda}) - \eta \varphi_0, \tag{33}$$

where ϑ, $\eta : T^*(\mathcal{M}^4) \to T(\mathcal{M}^4)$ are a pair of linear Poisson operators of the forms

$$\eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\vartheta = \begin{pmatrix} -\bar{f}_1 \Pi \bar{f}_1 & \begin{matrix} -\bar{f}_2 \Delta^{-1} \mathcal{E} \bar{f}_1 - \\ -\bar{f}_1 \Delta^{-1} \bar{f}_2 \end{matrix} & \begin{matrix} \mathcal{E} + \bar{f}_1 \Pi \bar{f}_1^* + \\ +\bar{f}_2 \Delta^{-1} \bar{f}_2^* + \\ +\bar{P} \end{matrix} & \bar{f}_1 \Delta^{-1} \mathcal{E} \bar{f}_2^* \\ \begin{matrix} -\bar{f}_1 \Delta^{-1} \bar{f}_2 - \\ -\bar{f}_2 \Delta^{-1} \mathcal{E} \bar{f}_1 \end{matrix} & -\bar{f}_2 \Pi \bar{f}_2 & \bar{f}_2 \Delta^{-1} \mathcal{E} \bar{f}_1^* & \begin{matrix} \mathcal{E} + \bar{f}_2 \Pi \bar{f}_2^* + \\ +\bar{f}_1 \Delta^{-1} \bar{f}_1^* + \\ +\bar{P} \end{matrix} \\ \begin{matrix} -\mathcal{E}^{-1} + \bar{f}_1^* \Pi \bar{f}_1 + \\ +\bar{f}_2^* \Delta^{-1} \mathcal{E} \bar{f}_2 - \bar{P} \end{matrix} & \bar{f}_1^* \Delta^{-1} \bar{f}_2 & -\bar{f}_1^* \Pi \bar{f}_1^* & \begin{matrix} -\bar{f}_1^* \Delta^{-1} \mathcal{E} \bar{f}_2^* - \\ -\bar{f}_2^* \Delta^{-1} \bar{f}_1^* \end{matrix} \\ \bar{f}_2^* \Delta^{-1} \bar{f}_1 & \begin{matrix} -\mathcal{E}^{-1} + \bar{f}_2^* \Pi \bar{f}_2 + \\ +\bar{f}_1^* \Delta^{-1} \mathcal{E} \bar{f}_1 - \bar{P} \end{matrix} & \begin{matrix} -\bar{f}_2^* \Delta^{-1} \bar{f}_1^* - \\ -\bar{f}_1^* \Delta^{-1} \mathcal{E} \bar{f}_2^* \end{matrix} & -\bar{f}_2^* \Pi \bar{f}_2^* \end{pmatrix}.$$

Here $\Delta = (\mathcal{E} - 1)$, $\Pi = \Delta^{-1}(\mathcal{E} + 1)$. Taking into account the equality

$$\varphi(n; \bar{\lambda}_i) = \left(\overline{\frac{d}{d\lambda} \operatorname{tr} S(n; \lambda) \Big|_{\lambda = \lambda_i}} \right) grad\, \lambda_i,$$

we find for every $i = \overline{1, N}$ that

$$\Lambda\, grad\, \lambda_i = \bar{\lambda}_i\, grad\, \lambda_i + \left(\overline{\frac{d}{d\lambda} \operatorname{tr} S(n; \lambda) \Big|_{\lambda = \lambda_i}} \right)^{-1} \varphi_0, \quad \Lambda = \eta^{-1} \vartheta,$$

where $\sigma_i := \left(\frac{d}{d\lambda} \operatorname{tr} S(n;\lambda)\Big|_{\lambda=\lambda_i} \right)^{-1} = \sum_{\chi=1}^{3} y_{\chi i} z_{\chi i}$ is invariant with respect to the vector fields $d/d\tau$ and d/dT.

Then on the invariant subspace $\mathcal{M}_N^4 \cap H_c \subset \mathcal{M}^4$ the gradients of the conservation laws $\gamma_m \in \mathcal{D}(\mathcal{M}^4)$, $m \in \mathbb{Z}_+$, take the forms

$$\varphi_0 = \sum_{i=1}^{N} grad\,\lambda_i, \quad \varphi_1 = \Lambda\varphi_0 = \sum_{i=1}^{N} \bar{\lambda}_i\, grad\,\lambda_i + \bar{J}_1 \sum_{i=1}^{N} grad\,\lambda_i, \quad \ldots,$$

$$\varphi_r = \Lambda\varphi_{r-1} = \sum_{i=1}^{N} \bar{\lambda}_i^r\, grad\,\lambda_i + \sum_{p=1}^{r} J_p \sum_{i=1}^{N} \bar{\lambda}_i^{r-p}\, grad\,\lambda_i, \quad \text{etc.}, \tag{34}$$

where

$$J_1 = \sum_{i=1}^{N} \sigma_i, \quad J_2 = \sum_{i=1}^{N} \lambda_i \sigma_i + J_1 \sum_{i=1}^{N} \sigma_i, \quad \ldots, J_r = \sum_{i=1}^{N} \lambda_i^r \sigma_i + \sum_{p=1}^{r} J_p \sum_{i=1}^{N} \lambda_i^{r-p} \sigma_i, \quad \text{etc.}$$

From the relationships (33) and (34) we obtain directly the explicit forms of the entries V_{13}, V_{23}, V_{31}, V_{31}, V_{32}, V_{33} on $\mathcal{M}_N^4 \cap H_c$ such as

$$V_{13} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \sum_{i=1}^{N} \frac{y_{1i} \tilde{z}_{3i}}{\lambda - \lambda_i}, \quad V_{23} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \sum_{i=1}^{N} \frac{y_{2i} \tilde{z}_{3i}}{\lambda - \lambda_i},$$
$$V_{31} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \sum_{i=1}^{N} \frac{y_{3i} \tilde{z}_{1i}}{\lambda - \lambda_i}, \quad V_{32} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \sum_{i=1}^{N} \frac{y_{3i} \tilde{z}_{2i}}{\lambda - \lambda_i},$$
$$V_{33} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \sum_{i=1}^{N} \frac{y_{3i} \tilde{z}_{3i}}{\lambda - \lambda_i},$$

where $1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} = \left(1 - \sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i}\right)^{-1}$.

The remaining entries of the reduced matrix $V_N := V|_{\mathcal{M}_N^4 \cap H_c}$ can be derived from the equation (32), considered on the level surfaces h_C, $C \in \mathbb{C}$, of the invariant function $1 - \rho$. On these surfaces the functions f_1^*, f_2^* satisfy the following equalities

$$f_1^* \left(1 - C + \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{1i} z_{1i}\right) + f_2^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{1i} z_{2i} = \sum_{i_1=1}^{N} \frac{1}{\lambda_i} y_{1i} z_{3i},$$
$$f_1^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{2i} z_{1i} + f_2^* \left(1 - C + \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{2i} z_{2i}\right) = \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{2i} z_{3i},$$
$$- f_1^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{3i} z_{1i} - f_2^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{3i} z_{2i} + \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{3i} z_{3i} = C.$$

Thus, the reduced matrix V_N on $\mathcal{M}_N^4 \cap H_c \cap h_C$, $C \in \mathbb{C}$, is written as

$$V_N = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \check{V}_N,$$

where

$$\check{V}_N = \sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i} \begin{pmatrix} y_{1i}\tilde{z}_{1i} & y_{1i}\tilde{z}_{2i} & y_{1i}\tilde{z}_{3i} \\ y_{2i}\tilde{z}_{1i} & y_{2i}\tilde{z}_{2i} & y_{2i}\tilde{z}_{3i} \\ y_{3i}\tilde{z}_{1i} & y_{3i}\tilde{z}_{2i} & y_{3i}\tilde{z}_{3i} \end{pmatrix} + \begin{pmatrix} -C & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The explicit form (31) of the monodromy matrix S_N on $\mathcal{M}_N^4 \cap H_c \cap h_C$, $C \in \mathbb{C}$, follows from the relationship

$$S_N = V_N A_N,$$

where the matrix $A_N := A_N(\mathcal{Y}, \mathcal{Z}; \lambda) = A[\mathbf{f}; \lambda]|_{\mathcal{M}_\mathcal{F} \cap h_C}$ is a projection of the matrix A on $\mathcal{M}_N^4 \cap H_c \cap h_C$. Thus,

$$S_N = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r}\right) \check{S}_N,$$

where the matrix $\check{S}_N$ has the form (31) when $|\lambda| > \max_{i=\overline{1,N}} |\lambda_i|$ and $\left(\sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i}\right) \neq 1$.

The relations (29) and (30) are derived from the monodromy matrix equation [6]

$$(\mathcal{E}S)A = AS$$

and compatibility conditions (20)-(21). □

Due to the equations (29) and (30) the functionals $\frac{1}{\alpha} \operatorname{tr} \check{S}_N^\alpha$, $\alpha \in \mathbb{N}$, are invariant with respect to the vector fields $d/d\tau$ and d/dT. Then the coefficients in the expansions of these functionals by poles appear to be conservation laws of the reduced upon $\mathcal{M}_\mathcal{F} \cap h_C$, $C \in \mathbb{C}$, vector fields given by the system (8), (9). The coefficients $\sigma_i, \hat{\sigma}_i, \tilde{\sigma}_i \in C^\infty(\mathbb{R}^{6N}; \mathbb{R})$, $i = \overline{1,N}$, in the expansions of the invariant functionals $\operatorname{tr} \check{S}_N$, $\frac{1}{2} \operatorname{tr} \check{S}_N^2$ and $\frac{1}{3} \operatorname{tr} \check{S}_N^3$ such that

$$\operatorname{tr} \check{S}_N = \sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i} - 3C + 1,$$

$$\frac{1}{2} \operatorname{tr} \check{S}_N^2 = \frac{1}{2} \sum_{i=1}^{N} \frac{\sigma_i^2}{(\lambda - \lambda_i)^2} + \sum_{i=1}^{N} \frac{\hat{\sigma}_i}{\lambda - \lambda_i} + \frac{1}{2}(3C^2 - 2C + 1),$$

$$\hat{\sigma}_i = \sum_{k=1,\, k\neq i}^{N} \frac{\operatorname{tr}(S_i S_k)}{\lambda_i - \lambda_k} + \operatorname{tr}(S_0 S_i) = \sum_{k=1,\, k\neq i}^{N} \frac{\left(\sum_{\chi_1=1}^{3} y_{\chi_1 i} z_{\chi_1 k}\right)\left(\sum_{\chi_2=1}^{3} y_{\chi_2 k} z_{\chi_2 i}\right)}{\lambda_i - \lambda_k}$$
$$- 3C(y_{1i}z_{1i} + y_{2i}z_{2i} + y_{3i}z_{3i}) + y_{3i}z_{3i},$$

and

$$\frac{1}{3}\operatorname{tr}\tilde{S}_N^3 = \frac{1}{3}\sum_{i=1}^{N}\frac{\sigma_i^3}{(\lambda-\lambda_i)^3} + \sum_{i=1}^{N}\frac{\sigma_i\hat{\sigma}_i}{(\lambda-\lambda_i)^2} + \sum_{i=1}^{N}\frac{\tilde{\sigma}_i}{\lambda-\lambda_i} - \frac{1}{3}(3C^3-3C^2+3C-1),$$

$$\tilde{\sigma}_i = \sum_{\substack{k,\ell=1,\\ k,\ell\neq i,\, k\neq \ell}}^{N}\frac{\operatorname{tr}(S_iS_kS_\ell)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_\ell)} + \sum_{k=1,\,k\neq i}^{N}\frac{\operatorname{tr}(S_iS_k)\,(\sigma_k-\sigma_i)}{(\lambda_i-\lambda_k)^2}$$

$$+\sum_{k=1,\,k\neq i}^{N}\frac{\operatorname{tr}(S_0(S_iS_k+S_kS_i))}{\lambda_i-\lambda_k} + \operatorname{tr}(S_0^2S_i)$$

$$= \sum_{\substack{k,\ell=1,\\ k,\ell\neq i,\, k\neq \ell}}^{N}\frac{\left(\sum_{\chi_1=1}^{3}y_{\chi_1 i}z_{\chi_1 \ell}\right)\left(\sum_{\chi_2=1}^{3}y_{\chi_2 \ell}z_{\chi_2 k}\right)\left(\sum_{\chi_3=1}^{3}y_{\chi_3 k}z_{\chi_3 i}\right)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_\ell)}$$

$$+\sum_{k=1,\,k\neq i}^{N}\frac{\left(\sum_{\chi_1=1}^{3}y_{\chi_1 i}z_{\chi_1 k}\right)\left(\sum_{\chi_2=1}^{3}y_{\chi_2 k}z_{\chi_2 i}\right)\sum_{\chi_3=1}^{3}(y_{\chi_3 i}z_{\chi_3 i}-y_{\chi_3 k}z_{\chi_3 k})}{(\lambda_i-\lambda_k)^2}$$

$$+\sum_{k=1,\,k\neq i}^{N}\frac{(-C(y_{1i}z_{1k}+y_{2i}z_{2k}+y_{3i}z_{3k})+y_{3i}z_{3k})\left(\sum_{\chi=1}^{3}y_{\chi k}z_{\chi i}\right)}{\lambda_i-\lambda_k}$$

$$+\sum_{k=1,\,k\neq i}^{N}\frac{(-C(y_{1k}z_{1i}+y_{2k}z_{2i}+y_{3k}z_{3i})+y_{3k}z_{3i})\left(\sum_{\chi=1}^{3}y_{\chi i}z_{\chi k}\right)}{\lambda_i-\lambda_k}$$

$$+(C^2y_{1i}z_{1i}+C^2y_{2i}z_{2i}+(1-C)^2y_{3i}z_{3i}),$$

are functionally independent on $\mathcal{M}_{\mathcal{F}}\bigcap h_C$, $C\in\mathbb{C}$. Being involutive with respect to the Poisson bracket $\{.,.\}_{\omega^{(2)}}$, the coefficients $\sigma_i,\hat{\sigma}_i,\tilde{\sigma}_i\in C^\infty(\mathbb{R}^{6N};\mathbb{R})$, $i=\overline{1,N}$, ensure the Liouville integrability of the vector fields $d/d\tau$ and d/dT on the finite-dimensional subspaces $\mathcal{M}_{\mathcal{F}}\bigcap h_C$, $C\in\mathbb{C}$ (see [1], [17]). The surfaces h_C, $C\in\mathbb{C}$, mentioned in Theorem 2, are determined by the conditions

$$\left(\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{3i_1}\left(1-C+\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{2i_2}\right)-\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{2i_1}z_{3i_1}\left(\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{1i_2}z_{2i_2}\right)\right)\sum_{i=1}^{N}\frac{1}{\lambda_{i_3}}y_{3i_3}z_{1i_3}$$

$$-\left(\left(1-C+\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{1i_1}\right)\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{3i_2}-\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{2i_1}\left(\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{1i_2}\right)\right)\sum_{i=1}^{N}\frac{1}{\lambda_{i_3}}y_{3i_3}z_{2i_3}$$

$$+\left(C-\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{3i_1}z_{3i_3}\right)\left(\left(1-C+\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{1i_2}z_{1i_2}\right)\right.$$

$$\left.\times\left(1-C+\sum_{i_3=1}^{N}\frac{1}{\lambda_{i_3}}y_{2i_3}z_{2i_3}\right)-\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{1i_2}z_{2i_2}\left(\sum_{i_3=1}^{N}\frac{1}{\lambda_{i_3}}y_{2i_3}z_{1i_3}\right)\right)=0,$$

when

$$\left(1-C+\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{1i_1}\right)\left(1-C+\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{2i_2}\right)$$

$$-\left(\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{2i_1}\right)\left(\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{1i_2}\right)\neq 0.$$

4 CONCLUSION

In the present paper by use of the method [2], [8], [9], [11], [22], [21] of reducing upon the special finite-dimensional invariant subspaces we have investigated the Bargmann type reduction of the Lax integrable two-dimensional generalization of the relativistic Toda lattice [10]. We have shown that the symplectic structure on the corresponding finite-dimensional invariant subspace can be found by means of the discrete analog of the Gelfand-Dikii relationship for the related Lagrangian function on a suitably extended phase space. This invariant subspace has been established to be diffeomorphic to the symplectic manifold smoothly embedded into space R^{6N}, $N\in\mathbb{N}$, with the canonical symplectic structure. The Lax-Liouville integrability of the reduced vector fields given by the system has been proven.

If $R=2$, for every $s\in\mathbb{N}$, $s\geq 2$, the evolutions of the vector-function $(f_1,f_2,f_1^*,f_2^*)^\top\in\mathcal{M}^4$, which are generated by the vector fields $d/dT_s := d/dt_s + d/d\tau_{s,1}$ and $d/dT_2 := d/dT$ and written out with taking into account the equalities

$$l^sf_1 = (d/d\tau + M_1^1)^sf_1,\quad l^sf_1^* = (-d/d\tau + M_1^{1*})^sf_1^*,$$

together with the relationship

$$dl_+^s/d\tau_{s,1} = [l_+^s, M_1^1]_+,$$

determine $(2+1)$-dimensional nonlinear dynamical system with the triple Lax type linearization. The symplectic finite-dimensional manifold described in the paper is a common invariant subspace of the vector fields $d/dT_s := d/dt_s + d/d\tau_{s,1}$, $s\in\mathbb{N}$, on which they are Hamiltonian and integrable by Liouville. Thus, it is interesting to investigate the possibility of applying the integration procedure, developed for the Liouville integrable finite-dimensional systems in [24], to the vector fields reduced upon this invariant subspace. The integration procedure [24] is based on the specially constructed Picard-Fuchs type differential-functional equations which generate the Hamiltonian-Jacobi transformations.

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Гентош О.Є. *Редукція Баргмана для деякого інтегровного за Лаксом двовимірного узагальнення релятивістського ланцюжка Тоди* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 155–171.

Досліджується можливість застосування методу редукування на скінченновимірні інваріантні підпростори, породжені власними значеннями асоційованої спектральної задачі, для деякого двовимірного узагальнення релятивістського ланцюжка Тоди з потрійною матричною лінеаризацією типу Лакса. Встановлено гамільтоновість та інтегровність за Лаксом-Ліувіллем заданих цією системою векторних полів на інваріантному підпросторі, пов'язаному з редукцією типу Баргмана.

Ключові слова і фрази: релятивістський ланцюжок Тоди, потрійна лінеаризація типу Лакса, інваріантна редукція, симплектична структура, інтегровність за Ліувіллем.

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GENERALIZED TYPES OF THE GROWTH OF DIRICHLET SERIES

Let Φ be a continuous function on $[\sigma_0, A)$ such that $\Phi(\sigma) \to +\infty$ as $\sigma \to A-0$, where $A \in (-\infty, +\infty]$. We establish a necessary and sufficient condition on a nonnegative sequence $\lambda = (\lambda_n)$, increasing to $+\infty$, under which the equality

$$\varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma,F)}{\Phi(\sigma)} = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma,F)}{\Phi(\sigma)}$$

holds for every Dirichlet series of the form $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s = \sigma + it$, which is absolutely convergent in the half-plane $\operatorname{Re} s < A$. Here $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$ and $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ are the maximum modulus and maximal term of this series respectively.

Key words and phrases: Dirichlet series, maximum modulus, maximal term, generalized type.

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INTRODUCTION

Let $\mathbb{N}_0$ be the set of all nonnegative integer numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, Λ be the class of all nonnegative sequences $\lambda = (\lambda_n)$, increasing to $+\infty$, $A \in (-\infty, +\infty]$, and Ω_A be the class of all continuous functions Φ on $[\sigma_0, A)$, such that

$$\forall x \in \mathbb{R}: \quad \lim_{\sigma\uparrow A}(x\sigma - \Phi(\sigma)) = -\infty. \tag{1}$$

Note that in the case $A < +\infty$ the condition (1) is equivalent to the condition $\Phi(\sigma) \to +\infty$, $\sigma \to A-0$, and in the case $A = +\infty$ this condition is equivalent to the condition $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$.

For a sequence $\lambda \in \Lambda$ let

$$\tau(\lambda) = \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n}.$$

Consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{2}$$

and put

$$E_1(F) = \left\{\sigma \in \mathbb{R} : \sum_{n=0}^{\infty} |a_n| e^{\sigma\lambda_n} < +\infty\right\}, \quad E_2(F) = \left\{\sigma \in \mathbb{R} : \lim_{n\to\infty} |a_n| e^{\sigma\lambda_n} = 0\right\},$$

$$\sigma_a(F) = \begin{cases} -\infty, & \text{if } E_1(F) = \varnothing, \\ \sup E_1(F), & \text{if } E_1(F) \neq \varnothing, \end{cases} \qquad \beta(F) = \begin{cases} -\infty, & \text{if } E_2(F) = \varnothing, \\ \sup E_2(F), & \text{if } E_2(F) \neq \varnothing \end{cases}$$

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($\sigma_a(F)$ is the abscissa of absolute convergence for the Dirichlet series (2)).

It is easy to show that

$$\beta(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

Also, it is well known (see, for example, [7, p. 114–115]), that

$$\sigma_a(F) \leq \beta(F) \leq \sigma_a(F) + \tau(\lambda)$$

and these inequalities are sharp (more precisely, for every $A, B \in \overline{\mathbb{R}}$ such that $A \leq B \leq A + \tau(\lambda)$ there exists [3] a Dirichlet series F of the form (2) such that $\sigma_a(F) = A$ and $\beta(F) = B$).

If $\sigma_a(F) > -\infty$, then for each $\sigma < \sigma_a(F)$ let $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$ be the maximum modulus of the series (2). If $\beta(F) > -\infty$, then for each $\sigma < \beta(F)$ let $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \in \mathbb{N}_0\}$ be the maximal term of this series. As is well known, in the case $\sigma_a(F) > -\infty$ we have $\mu(\sigma, F) \leq M(\sigma, F)$ for all $\sigma < \sigma_a(F)$.

By $\mathcal{D}_A(\lambda)$ we denote the class of all Dirichlet series of the form (2) such that $\sigma_a(F) \geq A$. Put $\mathcal{D}_A = \cup_{\lambda\in\Lambda}\mathcal{D}_A(\lambda)$. For $\Phi \in \Omega_A$ and $F \in \mathcal{D}_A$, the value

$$T_\Phi(F) = T_{\Phi,A}(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma,F)}{\Phi(\sigma)}$$

will be called Φ-*type* of the series F in the half-plane $\{s : \operatorname{Re} s < A\}$.

By $\mathcal{D}^*_A(\lambda)$ we denote the class of all Dirichlet series of the form (2) such that $\beta(F) \geq A$. Set $\mathcal{D}^*_A = \cup_{\lambda\in\Lambda}\mathcal{D}^*_A(\lambda)$. For $\Phi \in \Omega_A$ and $F \in \mathcal{D}^*_A$ we put

$$t_\Phi(F) = t_{\Phi,A}(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma,F)}{\Phi(\sigma)}.$$

If $F \in \mathcal{D}_A$, then $\mu(\sigma, F) \leq M(\sigma, F)$ for each $\sigma < A$, so $t_\Phi(F) \leq T_\Phi(F)$.

Note that $\mathcal{D}_A(\lambda) \subset \mathcal{D}^*_A(\lambda)$ for every sequence $\lambda \in \Lambda$. From what has been said above it follows that in the case $A < +\infty$ we have $\mathcal{D}_A(\lambda) = \mathcal{D}^*_A(\lambda)$ if and only if $\tau(\lambda) = 0$. Furthermore, $\mathcal{D}_{+\infty}(\lambda) = \mathcal{D}^*_{+\infty}(\lambda)$ if and only if $\tau(\lambda) < +\infty$. It is clear that $\mathcal{D}_A \subset \mathcal{D}^*_A$ and $\mathcal{D}_A \neq \mathcal{D}^*_A$.

The notion of Φ-type generalizes the classical notion of the type for entire Dirichlet series.

Let F be an entire Dirichlet series, i. e. $F \in \mathcal{D}_{+\infty}$, and ρ be a fixed positive number. Recall that

$$T(F) = \varlimsup_{\sigma\uparrow+\infty} \frac{\ln M(\sigma,F)}{e^{\rho\sigma}}$$

is called the *type* of the series F. If $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then the type of every entire Dirichlet series of the form (2) can be calculated (see, for example, [7, p. 178]) by the formula

$$T(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{e\rho}|a_n|^{\frac{\rho}{\lambda_n}}. \tag{3}$$

Let $\Phi \in \Omega_A$. The function

$$\widetilde{\Phi}(x) = \sup\{x\sigma - \Phi(\sigma) : \sigma \in [\sigma_0, A)\}, \quad x \in \mathbb{R},$$

is said to be *Young conjugate to* Φ (see, for example, [1, pp. 86–88]). The following properties of the function $\widetilde{\Phi}$ are well known (see also Lemmas 2 and 3 below): $\widetilde{\Phi}$ is convex on $\mathbb{R}$; if φ is the right-hand derivative of $\widetilde{\Phi}$, then $\widetilde{\Phi}(x) = x\varphi(x) - \Phi(\varphi(x))$, $x \in \mathbb{R}$, $\varphi(x) < A$ on $\mathbb{R}$ and

$\varphi(x) \nearrow A$ as $x \uparrow +\infty$; if $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$, then the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increase to A on $(x_0, +\infty)$. Since $\widetilde{\Phi}$ is convex on $\mathbb{R}$, $\widetilde{\Phi}$ is continuous on $\mathbb{R}$. Thus, the function $\overline{\Phi}$ is continuous on $(x_0, +\infty)$. Let $A_0 = \overline{\Phi}(x_0 + 0)$ and $\psi : (A_0, A) \to (x_0, +\infty)$ be the inverse function of $\overline{\Phi}$. Set $\psi(\sigma) = +\infty$ for $\sigma \in [A, +\infty]$. Let $F \in \mathcal{D}_A^*$ be a Dirichlet series of the form (2). Then $\beta(F) \geq A$, so that

$$\frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \geq A_0, \quad n \geq n_0.$$

Let $t > 0$ be a fixed number and $h(\sigma) = t\Phi(\sigma)$, $\sigma \in [\sigma_0, A)$. Then $\widetilde{h}(x) = t\widetilde{\Phi}(x/t)$, $x \in \mathbb{R}$, and hence $\overline{h}(x) = x\overline{\Phi}(x/t)$, $x \geq tx_0$. Using Lemma 5, given below, we obtain

$$t_\Phi(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{\psi\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}. \tag{4}$$

Therefore, for every Dirichlet series $F \in \mathcal{D}_A^*$ of the form (2) we have (4). Consequently, if $F \in \mathcal{D}_A$ is a Dirichlet series of the form (2) such that $T_\Phi(F) = t_\Phi(F)$, then Φ-type of this series can be calculated by the formula

$$T_\Phi(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{\psi\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}. \tag{5}$$

Note, that in the classical case, considered above ($A = +\infty$, $\Phi(\sigma) = e^{\rho\sigma}$), the formula (5) coincides with the formula (3). In this connection the following problem arises.

Problem 1. *Let $\lambda \in \Lambda$, $\Phi \in \Omega_A$. Find a necessary and sufficient condition on the sequence λ and the function Φ under which $T_\Phi(F) = t_\Phi(F)$ for every Dirichlet series $F \in \mathcal{D}_A$.*

In particular cases Problem 1 is solved in [2, 4, 5, 8, 6]. Denote by Ω_A^* the class of all function $\Phi \in \Omega_A$, convex on $[\sigma_0, A)$. If $\Phi \in \Omega_A^*$, then the one-sided derivatives Φ'_- and Φ'_+ are nondecreasing functions on $[\sigma_0, A)$ and $\Phi'_-(\sigma) \to +\infty$, $x \uparrow A$. Besides, using the definition of the function $\widetilde{\Phi}$ and Lemma 3, given below, it is easy to prove that

$$\Phi'_-(\varphi(x)) \leq x \leq \Phi'_+(\varphi(x)), \quad x > x_0 := \Phi'_+(\sigma_0). \tag{6}$$

The solution of Problem 1, in the case of the sequence $\lambda = (n)$ and an arbitrary function $\Phi \in \Omega_A^*$, was obtained practically in [2, 4] for $A = +\infty$ and in [5] for every $A \in (-\infty, +\infty]$ (actually, the growth of power series was investigated in [2, 4, 5]). We state a result from [5] in the following equivalent formulation.

Theorem A. *Let $\lambda = (n)$, $A \in (-\infty, +\infty]$, and $\Phi \in \Omega_A^*$. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the equality $T_\Phi(F) = t_\Phi(F)$ holds if and only if*

$$\ln \Phi'_+(\sigma) = o(\Phi(\sigma)), \quad \sigma \uparrow A.$$

Let $\Phi : [\sigma_0, A) \to \mathbb{R}$ be a continuously differentiable function from the class Ω_A^* such that Φ' is a positive function, increasing on $[\sigma_0, A)$. From (6) it follows that the restriction of the right-hand derivative φ of the function $\widetilde{\Phi}$ to $(x_0, +\infty)$ is the inverse function of Φ'. Put

$$\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}, \quad \sigma \in [\sigma_0, A).$$

(As is well known, the function Ψ is called the *Newton transform of* Φ.) It is easy to see that $\Psi(\varphi(x)) = \overline{\Phi}(x)$, $x \in [x_0, +\infty)$. For a sequence $\lambda \in \Lambda$, let $n_\lambda(x) = \sum_{\lambda_n \leq x} 1$ be its counting function. The next theorem was proved by M. M. Sheremeta [8].

Theorem B. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A^*$ be a twice continuously differentiable function on $[\sigma_0, A)$ such that $\Phi'(\sigma)/\Phi(\sigma) \nearrow +\infty$ and $\ln \Phi'(\sigma) = o(\Phi(\sigma))$ as $\sigma \uparrow A$. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the inequality $t_\Phi(F) \leq 1$ implies the inequality $T_\Phi(F) \leq 1$ if and only if*

$$\ln n_\lambda(x) = o(\Phi(\Psi(\varphi(x)))), \quad x \to +\infty. \tag{7}$$

Remark 1. *We can rewrite (7) in the form*

$$\ln n_\lambda(x) = o(\Phi(\overline{\Phi}(x))), \quad x \to +\infty.$$

Furthermore, as is easily seen, the condition (7) is equivalent to each of the conditions

$$\ln n_\lambda(\Phi'(\sigma)) = o(\Phi(\Psi(\sigma))), \quad \sigma \uparrow A;$$
$$\ln n = o(\Phi(\overline{\Phi}(\lambda_n))), \quad n \to \infty.$$

Remark 2. *The sufficiency of the condition (7) in Theorem B was proved in [8] only by the condition that $\Phi \in \Omega_A^*$ is a twice continuously differentiable function such that the function Φ'/Φ is nondecreasing on $[\sigma_0, A)$.*

Let $t \in (0, +\infty)$ be a fixed number. If Φ satisfy the conditions of Theorem B, then the function $t\Phi$ also satisfy these conditions. Applying Theorem B with $t\Phi$ instead of Φ and taking into account Remark 1, we see that $T_\Phi(F) = t_\Phi(F)$ for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ if an only if

$$\forall t > 0 : \quad \ln n = o(\Phi(\overline{\Phi}(\lambda_n/t))), \quad n \to \infty. \tag{8}$$

Note also that Theorem B does not imply Theorem A. In addition, Theorem B does not give the answer to the next question: whether the condition $\tau(\lambda) = 0$ is necessary in order that (3) holds for every entire Dirichlet series of the form (2)? Note, that the positive answer to this question was obtained in [6].

In connection with Theorem B the next problem arises.

Problem 2. *Let $T_0 \geq t_0 \geq 0$ be arbitrary constants, $\lambda \in \Lambda$, and $\Phi \in \Omega_A$. Find a necessary and sufficient condition on the sequence λ and the function Φ under which for every Dirichlet series $F \in \mathcal{D}_A$ such that $t_\Phi(F) = t_0$ the inequality $T_\Phi(F) \leq T_0$ holds.*

In this article we obtain the complete solutions of Problems 1 and 2.

1 THE STATEMENT OF MAIN RESULTS

For a sequence $\lambda \in \Lambda$, a function $\Phi \in \Omega_A$ and every $t_2 > t_1 > 0$ we put

$$\Delta(t_1, t_2) = \Delta_{\Phi,\lambda}(t_1, t_2) = \varlimsup_{n\to\infty} \frac{\ln n}{t_1\widetilde{\Phi}(\lambda_n/t_1) - t_2\widetilde{\Phi}(\lambda_n/t_2)}.$$

First we mention some properties of $\Delta(t_1, t_2)$.

If d is a fixed number, then for the function $\gamma(t) = t\widetilde{\Phi}(d/t)$, $t \in \mathbb{R}\backslash\{0\}$, we have

$$\gamma'_+(t) = \widetilde{\Phi}\left(\frac{d}{t}\right) - \frac{d}{t}\varphi\left(\frac{d}{t}\right) = -\Phi\left(\varphi\left(\frac{d}{t}\right)\right).$$

Hence,

$$t_1\widetilde{\Phi}\left(\frac{d}{t_1}\right) - t_2\widetilde{\Phi}\left(\frac{d}{t_2}\right) = \int_{t_1}^{t_2}\Phi\left(\varphi\left(\frac{d}{t}\right)\right)dt. \qquad (9)$$

Let $a > 0$ be a fixed number. Consider the function $y = \Delta(a,t)$, $t \in (a,+\infty)$. Using (9), Lemmas 2 and 6, given below, and taking into account that the function $\alpha(x) = \Phi(\varphi(x))$ is positive on $(x_0,+\infty)$, for every $t_2 > t_1 > a$ we obtain

$$0 \le y(t_2) \le y(t_1) \le \frac{t_2 - a}{t_1 - a}y(t_2).$$

It follows from this that the next three cases are possible: the function y is identically equal to 0; the function y is identically equal to $+\infty$; the function y is positive continuous nonincreasing on $(a,+\infty)$.

Let $b > 0$ be a fixed number. Consider the function $y = \Delta(t,b)$, $t \in (0,b)$. Using again Lemma 6, for every $0 < t_1 < t_2 < b$ we obtain

$$0 \le y(t_1) \le \frac{b - t_2}{b - t_1}y(t_2).$$

This implies that if $y(t_2) = 0$ for some $t_2 \in (0,b)$, then $y(t) = 0$ on $(0,t_2]$; if $y(t_1) = +\infty$ for some $t_1 \in (0,b)$, then $y(t) = +\infty$ on $[t_1,b)$; if the function y does not take the value 0 and $+\infty$ at some point $t \in (0,b)$, then this function increase at the point t.

Note also that the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0,+\infty)$, by Lemma 3, given below. Consequently, from (9), for every $d \ge 0$ and $t_2 > t_1 > 0$, we have

$$(t_2 - t_1)\Phi\left(\varphi\left(\frac{d}{t_2}\right)\right) \le t_1\widetilde{\Phi}\left(\frac{d}{t_1}\right) - t_2\widetilde{\Phi}\left(\frac{d}{t_2}\right) \le (t_2 - t_1)\Phi\left(\varphi\left(\frac{d}{t_1}\right)\right). \qquad (10)$$

The solution of Problem 1 is contained in the following theorem.

Theorem 1. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, and $\Phi \in \Omega_A$. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the equality $T_\Phi(F) = t_\Phi(F)$ holds if and only if*

$$\forall t > 0: \quad \ln n = o(\Phi(\varphi(\lambda_n/t))). \qquad (11)$$

Remark 3. *The conditions* (8) *and* (11) *are equivalent for every function $\Phi \in \Omega_A^*$. This fact follows from the inequalities*

$$(1-q)\Phi(\varphi(qx)) \le \Phi(\overline{\Phi}(x)) < \Phi(\varphi(x)), \qquad (12)$$

which hold for every fixed $q \in (0,1)$ and all large enough x (see Lemma 8 below).

Note also that if $F \in \mathcal{D}_A(\lambda)$ and $t_\Phi(F) = +\infty$, then $T_\Phi(F) = +\infty$, by the inequality $\mu(\sigma,F) \le M(\sigma,F)$, $\sigma < A$, so that $T_\Phi(F) = t_\Phi(F)$. In this connection, the next theorem makes more precise Theorem 1 in the part of the sufficiency of (11).

Theorem 2. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, and $\Phi \in \Omega_A$. If the condition* (11) *holds, then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) < +\infty$ belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F) = t_\Phi(F)$.*

The solution of Problem 2 is contained in the following theorem.

Theorem 3. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, and $T_0 \ge t_0 \ge 0$ be arbitrary constants. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $t_\Phi(F) = t_0$ the inequality $T_\Phi(F) \le T_0$ holds if and only if*

$$\forall T > T_0\ \exists c \in (t_0,T): \quad \Delta(c,T) < 1. \qquad (13)$$

By Theorem 3, for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the inequality $t_\Phi(F) \le 1$ implies the inequality $T_\Phi(F) \le 1$ if and only if

$$\forall T > 1\ \exists c \in (1,T): \quad \Delta(c,T) < 1. \qquad (14)$$

If $A = +\infty$ and $\Phi(\sigma) = \sigma\ln\sigma$, $\sigma \ge e$, then, as is easy to show, the condition (14) becomes

$$\varlimsup_{n\to\infty}\frac{\ln\ln n}{\lambda_n} < 1,$$

but the condition (7) from Theorem B takes the form

$$\ln n = o(e^{\lambda_n}), \quad n \to \infty.$$

Hence, generally, the condition (14) does not coincide with the condition (7).

In the part of the sufficiency of (13) the Theorem 3 can be made more precise.

Theorem 4. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, and $T_0 \ge t_0 \ge 0$ be arbitrary constants. If the condition* (13) *holds, then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) = t_0$ belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F) \le T_0$.*

Theorems 3 and 4 follow immediately from Theorems 5 and 6, given below, respectively.

Theorem 5. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, and $T_0 > t_0 \ge 0$ be arbitrary constants. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $t_\Phi(F) = t_0$ the inequality $T_\Phi(F) < T_0$ holds if and only if*

$$\exists c \in (t_0,T_0): \quad \Delta(c,T_0) < 1. \qquad (15)$$

Theorem 6. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, and $T_0 > t_0 \ge 0$ be arbitrary constants. If the condition* (15) *holds, then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) = t_0$ belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F) < T_0$.*

Theorem 6 follows from the next more general result.

Theorem 7. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, and $\Phi,\Gamma \in \Omega_A$. If*

$$\sum_{n=0}^{\infty}\frac{1}{e^{\widetilde{\Phi}(\lambda_n)-\widetilde{\Gamma}(\lambda_n)}} < +\infty, \qquad (16)$$

then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $\ln\mu(\sigma,F) \le \Phi(\sigma)$, $\sigma \in [\sigma_1,A)$, belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $\ln M(\sigma,F) \le \Gamma(\sigma)$, $\sigma \in [\sigma_2,A)$.

2 AUXILIARY RESULTS

Denote by X the class of all functions $h:\mathbb{R}\to\overline{\mathbb{R}}$. Suppose $h\in X$ and let $\widetilde{h}\in X$ be the Young conjugate function to h, i. e.

$$\widetilde{h}(\sigma)=\sup\{\sigma x-h(x): x\in\mathbb{R}\},\quad \sigma\in\mathbb{R}.$$

It is clear that if $h,g\in X$ and $h(x)\ge g(x)$ for all $x\in\mathbb{R}$, then $\widetilde{h}(\sigma)\le\widetilde{g}(\sigma)$ for all $\sigma\in\mathbb{R}$.

Let $h\in X$. Then $\widetilde{\widetilde{h}}(x)\le h(x)$ for each $x\in\mathbb{R}$, where $\widetilde{\widetilde{h}}$ is the Young conjugate function to $\widetilde{h}$. Indeed, the definition of $\widetilde{h}$ implies that for every $\sigma,x\in\mathbb{R}$ the inequality $\sigma x-h(x)\le\widetilde{h}(\sigma)$ holds. Then $x\sigma-\widetilde{h}(\sigma)\le h(x)$ for every $\sigma,x\in\mathbb{R}$. From this it follows that $\widetilde{\widetilde{h}}(x)\le h(x)$ for each $x\in\mathbb{R}$.

Lemma 1. *Let $h,g\in X$. Then the following conditions are equivalent:*
(i) $\widetilde{h}(\sigma)\le g(\sigma)$ for all $\sigma\in\mathbb{R}$;
(ii) $h(x)\ge\widetilde{g}(x)$ for all $x\in\mathbb{R}$.

Proof. If the condition (i) holds, then $\widetilde{\widetilde{h}}(x)\ge\widetilde{g}(x)$ for each $x\in\mathbb{R}$. Since $\widetilde{\widetilde{h}}(x)\le h(x)$ for all $x\in\mathbb{R}$, from this it follows (ii).

If the condition (ii) holds, then $\widetilde{h}(\sigma)\le\widetilde{\widetilde{g}}(\sigma)$ for each $\sigma\in\mathbb{R}$. Since $\widetilde{\widetilde{g}}(\sigma)\le g(\sigma)$ for all $\sigma\in\mathbb{R}$, from this it follows (i). □

Lemma 2. *Let $h\in X$. Then $\widetilde{h}$ is a convex function on $\mathbb{R}$, i. e. for every $x_1,x_2,x_3\in\mathbb{R}$ such that $x_1\le x_2\le x_3$ we have*

$$\widetilde{h}(x_2)(x_3-x_1)\le\widetilde{h}(x_1)(x_3-x_2)+\widetilde{h}(x_3)(x_2-x_1).\tag{17}$$

Proof. For each $t\in\mathbb{R}$ we have

$$(tx_2-h(t))(x_3-x_1)=(tx_1-h(t))(x_3-x_2)+(tx_3-h(t))(x_2-x_1).$$

From this equality and the definition of $\widetilde{h}$ we have (17). □

For a function $h\in X$ we put $D_h=\{x\in\mathbb{R}: h(x)<+\infty\}$. It is clear that in the definition of $\widetilde{h}(\sigma)$ we can take the supremum by all $x\in D_h$ instead the supremum by all $x\in\mathbb{R}$.

Let $A\in(-\infty,+\infty]$ and $\Phi:[\sigma_0,A)\to\mathbb{R}$ be a function from the class Ω_A. We assume that the function Φ belong to the class X, setting $\Phi(\sigma)=+\infty$ for every $\sigma\notin[\sigma_0,+\infty)$ (then $D_\Phi=[\sigma_0,+\infty)$). Fix some $x\in\mathbb{R}$ and set

$$y(\sigma)=x\sigma-\Phi(\sigma),\quad \sigma\in[\sigma_0,A).$$

The function y is continuous on $[\sigma_0,A)$. In addition, by (1), $y(\sigma)\to-\infty$ as $\sigma\uparrow A$. Hence, this function assumes its supremum on $[\sigma_0,A)$, i. e.

$$\widetilde{\Phi}(x)=\max_{\sigma\ge\sigma_0}y(\sigma).$$

Consider the set

$$S(x)=\{\sigma\ge\sigma_0: y(\sigma)=\widetilde{\Phi}(x)\}.$$

From what has been said it follows that the set $S(x)$ is nonempty and bounded. Let $\varphi(x)=\sup S(x)$. Then $\varphi(x)\in S(x)$, i. e. $\max S(x)$ exists and $\varphi(x)=\max S(x)$. Indeed, if we assume that $\varphi(x)\notin S(x)$, then the set $S(x)$ is infinite and $\sigma<\varphi(x)$ for every $\sigma\in S(x)$. Let (σ_n) be a sequence of points in $S(x)$, increasing to $\varphi(x)$. For every $n\in\mathbb{N}_0$ we have $y(\sigma_n)=\widetilde{\Phi}(x)$. Letting n to ∞ and using the continuity of the function Φ, we obtain $y(\varphi(x))=\widetilde{\Phi}(x)$, i. e. $\varphi(x)\in S(x)$, but this contradicts the assumption that $\varphi(x)\notin S(x)$. Hence, $\max S(x)$ exists and $\varphi(x)=\max S(x)$.

Lemma 3. *Let $A\in(-\infty,+\infty]$, $\Phi\in\Omega_A$, and $\varphi(x)=\max\{\sigma\in[\sigma_0,A): x\sigma-\Phi(\sigma)=\widetilde{\Phi}(x)\}$, $x\in\mathbb{R}$. Then:*
(i) the function φ is nondecreasing on $\mathbb{R}$;
(ii) the function φ is continuous from the right on $\mathbb{R}$;
(iii) $\varphi(x)\to A$, $x\to+\infty$;
(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at every point $x\in\mathbb{R}$;
(v) if $x_0=\inf\{x>0:\Phi(\varphi(x))>0\}$, then the function $\overline{\Phi}(x)=\widetilde{\Phi}(x)/x$ increase to A on $(x_0,+\infty)$;
(vi) the function $\alpha(x)=\Phi(\varphi(x))$ is nondecreasing on $[0,+\infty)$.

Proof. (i) Let $x_1<x_2$. Since $x_j\varphi(x_j)-\Phi(\varphi(x_j))=\widetilde{\Phi}(x_j)$, $j\in\{1,2\}$, the definition of $\widetilde{\Phi}$ implies the following inequalities

$$x_1\varphi(x_1)-\Phi(\varphi(x_1))\ge x_1\varphi(x_2)-\Phi(\varphi(x_2)),\quad x_2\varphi(x_2)-\Phi(\varphi(x_2))\ge x_2\varphi(x_1)-\Phi(\varphi(x_1)).$$

Adding these inequalities, we obtain $(\varphi(x_2)-\varphi(x_1))(x_2-x_1)\ge0$. From this it follows that $\varphi(x_1)\le\varphi(x_2)$.

(ii) Let $x_0\in\mathbb{R}$ be a fixed point. By (i) it follows that the right-hand limit $\varphi(x_0+0)$ exists and $\varphi(x_0+0)\ge\varphi(x_0)$. Let us prove that $\varphi(x_0+0)=\varphi(x_0)$, i. e. that φ is continuous from the right at the point x_0. Indeed, the definition of $\widetilde{\Phi}$ implies the inequality

$$x\varphi(x_0)-\Phi(\varphi(x_0))\le x\varphi(x)-\Phi(\varphi(x)).$$

Letting x to x_0 from the right, we obtain $\widetilde{\Phi}(x_0)\le x_0\varphi(x_0+0)-\Phi(\varphi(x_0+0))$. On the other hand, $\widetilde{\Phi}(x_0)\ge x_0\varphi(x_0+0)-\Phi(\varphi(x_0+0))$. Hence, $\widetilde{\Phi}(x_0)=x_0\varphi(x_0+0)-\Phi(\varphi(x_0+0))$. Then from the definition of φ we obtain $\varphi(x_0+0)\le\varphi(x_0)$ and thus $\varphi(x_0+0)=\varphi(x_0)$.

(iii) Suppose the contrary, that is $\varphi(+\infty)=B<A$. Let $C\in(B,A)$. Using the definition of the function $\widetilde{\Phi}$, we have

$$xC-\Phi(C)\le x\varphi(x)-\Phi(\varphi(x))$$

for every $x\in\mathbb{R}$. This implies that

$$x(C-\varphi(x))\le\Phi(C)-\Phi(\varphi(x)).$$

Letting x to $+\infty$, we obtain $+\infty\le\Phi(C)-\Phi(B)$, but this is impossible.

(iv) Let $x\in\mathbb{R}$ be a fixed point and $h>0$. From the definition of the function $\widetilde{\Phi}$ we have

$$\frac{\widetilde{\Phi}(x+h)-\widetilde{\Phi}(x)}{h}\ge\frac{(x+h)\varphi(x)-\Phi(\varphi(x))-\widetilde{\Phi}(x)}{h}=\varphi(x),$$
$$\frac{\widetilde{\Phi}(x+h)-\widetilde{\Phi}(x)}{h}\le\frac{\widetilde{\Phi}(x+h)-(x\varphi(x+h)-\Phi(\varphi(x+h)))}{h}=\varphi(x+h).$$

Hence,

$$\varphi(x) \le \frac{\widetilde{\Phi}(x+h) - \widetilde{\Phi}(x)}{h} \le \varphi(x+h).$$

Letting h to 0 and using (ii), we see that the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$.

(v) Since $x\varphi(x) - \widetilde{\Phi}(x) = \Phi(\varphi(x)) > 0$ for $x > x_0$,

$$(\overline{\Phi}(x))'_+ = \frac{x\varphi(x) - \widetilde{\Phi}(x)}{x^2} > 0, \quad x > x_0.$$

Hence, the function $\overline{\Phi}(x)$ increase on $(x_0, +\infty)$. Furthermore, the inequality $x\varphi(x) - \widetilde{\Phi}(x) > 0$, $x > x_0$, implies that $\overline{\Phi}(x) < \varphi(x) < A$, $x > x_0$. On the other hand, for every fixed x_1 and each $x \ge x_1$ we have

$$\widetilde{\Phi}(x) = \widetilde{\Phi}(x_1) + \int_{x_1}^{x} \varphi(t)dt \ge \widetilde{\Phi}(x_1) + (x - x_1)\varphi(x_1).$$

From this it follows that

$$\varliminf_{x\to+\infty} \overline{\Phi}(x) \ge \varphi(x_1).$$

Letting x_1 to $+\infty$, we see that $\overline{\Phi}(x) \to A$, $x \to +\infty$.

(vi) Let $x_2 > x_1 \ge 0$. Then

$$\begin{aligned}\alpha(x_2) - \alpha(x_1) &= x_2\varphi(x_2) - x_1\varphi(x_1) + \widetilde{\Phi}(x_1) - \widetilde{\Phi}(x_2) \ge x_2\varphi(x_2) - x_1\varphi(x_1) + (x_1 - x_2)\varphi(x_2) \\ &= x_1(\varphi(x_2) - \varphi(x_1)) \ge 0.\end{aligned}$$

Therefore, the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$. □

Lemma 4. *Let $A \in (-\infty, +\infty]$, $\Phi_1, \Phi_2 \in \Omega_A$, and $\Phi_1(\sigma) = \Phi_2(\sigma)$ for all $\sigma \in [\sigma_0, A)$. Then $\widetilde{\Phi}_1(x) = \widetilde{\Phi}_2(x)$ for each $x \ge x_0$.*

Proof. For $j \in \{1,2\}$ let $D_{\Phi_j} = [\sigma_j, A)$ and

$$\varphi_j(x) = \max\{\sigma \in [\sigma_j, A) : x\sigma - \Phi_j(\sigma) = \widetilde{\Phi}_j(x)\}, \quad x \in \mathbb{R}.$$

Lemma 3 implies that $\min\{\varphi_1(x), \varphi_2(x)\} \ge \max\{\sigma_0, \sigma_1, \sigma_2\}$ for all $x \ge x_0$. Then for every $x \ge x_0$ we get

$$\widetilde{\Phi}_1(x) = x\varphi_1(x) - \Phi_1(\varphi_1(x)) = x\varphi_1(x) - \Phi_2(\varphi_1(x)) \le \max_{\sigma\ge\sigma_2}(x\sigma - \Phi_2(\sigma)) = \widetilde{\Phi}_2(x),$$
$$\widetilde{\Phi}_2(x) = x\varphi_2(x) - \Phi_2(\varphi_2(x)) = x\varphi_2(x) - \Phi_1(\varphi_2(x)) \le \max_{\sigma\ge\sigma_1}(x\sigma - \Phi_1(\sigma)) = \widetilde{\Phi}_1(x),$$

and, hence, $\widetilde{\Phi}_1(x) = \widetilde{\Phi}_2(x)$. □

Lemma 5. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $F \in \mathcal{D}^*_A$ be a Dirichlet series of the form* (2). *Then $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$ if and only if $\ln|a_n| \le -\widetilde{\Phi}(\lambda_n)$ for all $n \ge n_0$.*

Proof. Suppose that $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$. We set $\Psi(\sigma) = \Phi(\sigma)$ for every $\sigma \in [\sigma_0, A)$ and $\Psi(\sigma) = +\infty$ for every $\sigma \notin [\sigma_0, A)$. Let $h \in X$ be the function such that $h(\lambda_n) = -\ln|a_n|$ for all $n \in \mathbb{N}_0$ and $h(x) = +\infty$ for all $x \in \mathbb{R}\backslash\{\lambda_0, \lambda_1, \dots\}$. Then $\ln \mu(\sigma, F) = \widetilde{h}(\sigma)$ for $\sigma < \beta(F)$. Consequently, $\widetilde{h}(\sigma) \le \Psi(\sigma)$ for each $\sigma \in \mathbb{R}$. By Lemma 1, $h(x) \ge \widetilde{\Psi}(x)$, $x \in \mathbb{R}$. Therefore, using Lemma 4, we have $\ln|a_n| = -h(\lambda_n) \le -\widetilde{\Psi}(\lambda_n) = -\widetilde{\Phi}(\lambda_n)$ for all $n \ge n_0$.

Now suppose that $\ln|a_n| \le -\widetilde{\Phi}(\lambda_n)$ for all $n \ge n_0$. If the function $\mu(\sigma, F)$ is bounded on $(-\infty, A)$, then, obviously, $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$. If the function $\mu(\sigma, F)$ is unbounded on $(-\infty, A)$, then we consider, along with F, the Dirichlet series

$$G(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{18}$$

such that $b_n = 0$ for $n < n_0$ and $b_n = a_n$ for $n \ge n_0$. It is easy to show that $\mu(\sigma, F) = \mu(\sigma, G)$ for each $\sigma \in [\sigma_0, A)$. Besides, $\ln|b_n| \le -\widetilde{\Phi}(\lambda_n)$ for all $n \in \mathbb{N}_0$. Hence, by Lemma 1, we have $\ln \mu(\sigma, G) \le \Phi(\sigma)$, $\sigma < A$. This implies that $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$. □

Lemma 6. *Let Ψ be a function, convex on $\mathbb{R}$, and $x_0 \ge 0$. Then for all $t_1, t_2, t_3 \in \mathbb{R}$ such that $t_3 > t_2 > t_1 > 0$ we have*

$$t_1\Psi\left(\frac{x_0}{t_1}\right) - t_2\Psi\left(\frac{x_0}{t_2}\right) \ge \frac{t_2 - t_1}{t_3 - t_1}\left(t_1\Psi\left(\frac{x_0}{t_1}\right) - t_3\Psi\left(\frac{x_0}{t_3}\right)\right),$$
$$t_2\Psi\left(\frac{x_0}{t_2}\right) - t_3\Psi\left(\frac{x_0}{t_3}\right) \le \frac{t_3 - t_2}{t_3 - t_1}\left(t_1\Psi\left(\frac{x_0}{t_1}\right) - t_3\Psi\left(\frac{x_0}{t_3}\right)\right).$$

Proof. Since Ψ is convex on $\mathbb{R}$, for every $t_1, t_2, t_3 \in \mathbb{R}$ such that $t_3 > t_2 > t_1 > 0$ we have the following inequality

$$\Psi\left(\frac{x_0}{t_2}\right)\left(\frac{x_0}{t_1} - \frac{x_0}{t_3}\right) \le \Psi\left(\frac{x_0}{t_1}\right)\left(\frac{x_0}{t_2} - \frac{x_0}{t_3}\right) + \Psi\left(\frac{x_0}{t_3}\right)\left(\frac{x_0}{t_1} - \frac{x_0}{t_2}\right).$$

Multiplying this inequality by $t_1t_2t_3$, we obtain

$$\Psi\left(\frac{x_0}{t_2}\right)t_2(t_3 - t_1) \le \Psi\left(\frac{x_0}{t_1}\right)t_1(t_3 - t_2) + \Psi\left(\frac{x_0}{t_3}\right)t_3(t_2 - t_1).$$

From this it follows that

$$\begin{aligned}\Psi\left(\frac{x_0}{t_1}\right)t_1(t_3 - t_1) - \Psi\left(\frac{x_0}{t_2}\right)t_2(t_3 - t_1) &\ge \Psi\left(\frac{x_0}{t_1}\right)t_1(t_3 - t_1) - \Psi\left(\frac{x_0}{t_1}\right)t_1(t_3 - t_2) \\ &- \Psi\left(\frac{x_0}{t_3}\right)t_3(t_2 - t_1) = \Psi\left(\frac{x_0}{t_1}\right)t_1(t_2 - t_1) - \Psi\left(\frac{x_0}{t_3}\right)t_3(t_2 - t_1), \\ \Psi\left(\frac{x_0}{t_2}\right)t_2(t_3 - t_1) - \Psi\left(\frac{x_0}{t_3}\right)t_3(t_3 - t_1) &\le \Psi\left(\frac{x_0}{t_1}\right)t_1(t_3 - t_2) + \Psi\left(\frac{x_0}{t_3}\right)t_3(t_2 - t_1) \\ &- \Psi\left(\frac{x_0}{t_3}\right)t_3(t_3 - t_1) = \Psi\left(\frac{x_0}{t_1}\right)t_1(t_3 - t_2) - \Psi\left(\frac{x_0}{t_3}\right)t_3(t_3 - t_2).\end{aligned}$$

Lemma 6 is proved. □

We note, that some of the above properties of the Young conjugate functions are well known (see, for examle, [1, § 3.2]).

Lemma 7. *Let (x_n) be a positive sequence such that*

$$\varlimsup_{n\to\infty} \frac{\ln n}{x_n} = \delta \ge 1.$$

Then, for every $q \in (0,1)$, the set $E(q) = \{n \in \mathbb{N}_0 : \ln n \ge qx_n \wedge x_{[n/2]} \ge qx_n\}$ is unbounded.

Proof. If $\delta = +\infty$, then, setting $m_k = \min\{n \in \mathbb{N}_0 : \ln n \geq (k+1)x_n\}$, we see that $m_k \in E(q)$ for every $k \in \mathbb{N}_0$. If $\delta < +\infty$, then, for some increasing sequence (p_k) of nonnegative integers, we have $\ln p_k \sim \delta x_{p_k}$, $k \to \infty$. Therefore,

$$\varlimsup_{k\to\infty} \frac{x_{p_k}}{x_{[p_k/2]}} = \frac{1}{\delta} \varlimsup_{k\to\infty} \frac{\ln p_k}{x_{[p_k/2]}} = \frac{1}{\delta} \varlimsup_{k\to\infty} \frac{\ln[p_k/2]}{x_{[p_k/2]}} \leq \frac{1}{\delta} \varlimsup_{n\to\infty} \frac{\ln n}{x_n} = 1.$$

It is clear that $p_k \in E_q$ for all $k \geq k_0(q)$. □

Theorem 8. *Let $A \in (-\infty, +\infty]$, $\lambda \in \Lambda$ be a sequence such that $\tau(\lambda) > 0$ in the case $A < +\infty$ and $\tau(\lambda) = +\infty$ in the case $A = +\infty$, and $G \in \mathcal{D}_A^*(\lambda) \backslash \mathcal{D}_A(\lambda)$ be a Dirichlet series of the form* (18) *such that $b_n \geq 0$, $n \in \mathbb{N}_0$. Then for every positive on $(-\infty, A)$ function l there exists a Dirichlet series $F \in \mathcal{D}(\lambda)$ of the form* (2) *such that either $a_n = b_n$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $M(\sigma, F) = F(\sigma) \geq l(\sigma)$ for all $\sigma \in [\sigma_0, A)$.*

Proof. We may assume without loss of generality that the function l is nondecreasing on $(-\infty, A)$.

Since $G \in \mathcal{D}_A^*(\lambda) \backslash \mathcal{D}_A(\lambda)$, we have $\beta(G) \geq A$ and $\sigma_a(G) < A$. The inequality $\beta(G) \geq A$ implies that there exists a sequence (η_n), increasing to A, such that

$$\frac{1}{\lambda_n} \ln \frac{1}{b_n} \geq \eta_n, \quad n \in \mathbb{N}_0.$$

Then $b_n \leq e^{-\eta_n \lambda_n}$, $n \in \mathbb{N}_0$. Since $\sigma_a(G) < A$, for all $\sigma \in (\sigma_a(G), A)$ and every $m \in \mathbb{N}_0$ we have

$$\sum_{n \geq m} b_n e^{\sigma \lambda_n} = +\infty.$$

Fix some sequence (σ_n), increasing to A. We choose a sequence (m_k) of nonnegative integers to be so rapidly increasing that the inequalities

$$\eta_{m_k} \geq \sigma_k, \quad e^{(\sigma_k - \sigma_{k+1})\lambda_{m_{k+1}}}(l(\sigma_{k+2}) + 1) < \frac{1}{(k+1)^2}, \quad \sum_{n=m_k}^{m_{k+1}-1} b_n e^{\sigma_k \lambda_n} \geq l(\sigma_{k+1})$$

hold for every $k \in \mathbb{N}_0$. Put

$$p_k = \min\left\{ p \geq m_k : \sum_{n=m_k}^{p} b_n e^{\sigma_k \lambda_n} \geq l(\sigma_{k+1}) \right\}, \quad k \in \mathbb{N}_0.$$

Note that $m_k \leq p_k \leq m_{k+1} - 1$ and

$$l(\sigma_{k+1}) \leq \sum_{n=m_k}^{p_k} b_n e^{\sigma_k \lambda_n} < l(\sigma_{k+1}) + b_{p_k} e^{\sigma_k \lambda_{p_k}} \leq l(\sigma_{k+1}) + e^{(\sigma_k - \eta_{p_k})\lambda_{p_k}} \leq l(\sigma_{k+1}) + 1.$$

Let $n \in \mathbb{N}_0$. If $n \in [m_k, p_k]$ for some $k \in \mathbb{N}_0$, then we put $a_n = b_n$. If $n \notin [m_k, p_k]$ for every $k \in \mathbb{N}_0$, then let $a_n = 0$. Consider the Dirichlet series F of the form (2) and let us prove that $\sigma_a(G) \geq A$. Indeed, for every fixed $j \in \mathbb{N}_0$ we have

$$\begin{aligned} \sum_{n \geq m_{j+1}} a_n e^{\sigma_j \lambda_n} &= \sum_{k \geq j+1} \sum_{n=m_k}^{p_k} b_n e^{\sigma_j \lambda_n} = \sum_{k \geq j+1} \sum_{n=m_k}^{p_k} b_n e^{\sigma_k \lambda_n} e^{(\sigma_j - \sigma_k)\lambda_n} \\ &\leq \sum_{k \geq j+1} e^{(\sigma_j - \sigma_k)\lambda_{m_k}} \sum_{n=m_k}^{p_k} b_n e^{\sigma_k \lambda_n} \\ &\leq \sum_{k \geq j+1} e^{(\sigma_{k-1} - \sigma_k)\lambda_{m_k}}(l(\sigma_{k+1}) + 1) < \sum_{k \geq j+1} \frac{1}{k^2} < +\infty, \end{aligned}$$

so that $\sigma_a(F) \geq A$. Moreover, if $\sigma \in [\sigma_0, A)$, then $\sigma \in [\sigma_k, \sigma_{k+1})$ for some $k \in \mathbb{N}_0$ and therefore

$$F(\sigma) \geq \sum_{n=m_k}^{p_k} a_n e^{\sigma \lambda_n} = \sum_{n=m_k}^{p_k} b_n e^{\sigma \lambda_n} \geq \sum_{n=m_k}^{p_k} b_n e^{\sigma_k \lambda_n} \geq l(\sigma_{k+1}) \geq l(\sigma).$$

Theorem 8 is proved. □

Lemma 8. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A^*$, and $q \in (0,1)$. Then the inequalities* (12) *hold for all $x \geq x_0$.*

Proof. If $\Phi \in \Omega_A^*$, then the function Φ is increasing on $[\sigma_1, A)$. Since

$$\overline{\Phi}(x) = \varphi(x) - \frac{\Phi(\varphi(x))}{x} < \varphi(x), \quad x > x_1,$$

we have $\Phi(\overline{\Phi}(x)) < \Phi(\varphi(x))$, $x > x_2$, i.e. the right of the inequalities (12) holds.

Further, using the convexity of the function Φ and the inequalities (6), we have

$$\Phi(\varphi(x)) - \Phi(\varphi(qx)) \leq (\varphi(x) - \varphi(qx))\Phi'_{-}(\varphi(x)) \leq (\varphi(x) - \varphi(qx))x, \quad x > x_3,$$

and, hence, for all $x > x_4$ we obtain

$$\begin{aligned} \Phi(\varphi(qx)) - \Phi(\overline{\Phi}(x)) &\leq (\varphi(qx) - \overline{\Phi}(x))\Phi'_{-}(\varphi(qx)) \leq \left(\varphi(qx) - \varphi(x) + \frac{\Phi(\varphi(x))}{x} \right) qx \\ &\leq \left(\frac{\Phi(\varphi(qx)) - \Phi(\varphi(x))}{x} + \frac{\Phi(\varphi(x))}{x} \right) qx = q\Phi(\varphi(qx)). \end{aligned}$$

This implies the left of the inequalities (12). □

3 The proofs of main results

Proof of Theorem 7. Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, and $\Phi, \Gamma \in \Omega_A$ be functions that satisfy (16).

Consider a Dirichlet series $F \in \mathcal{D}_A^*(\lambda)$ of the form (2) such that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$, $\sigma \in [\sigma_1, A)$. By Lemma 5 we have $\ln|a_n| \leq -\widetilde{\Phi}(\lambda_n)$, $n \geq n_1$.

Fix $n_2 \geq n_1$ such that

$$\sum_{n \geq n_2} \frac{1}{e^{\widetilde{\Phi}(\lambda_n) - \widetilde{\Gamma}(\lambda_n)}} \leq \frac{1}{2}.$$

Then for all $\sigma \in [\sigma_2, A)$ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| e^{\sigma \lambda_n} &= \sum_{n < n_2} |a_n| e^{\sigma \lambda_n} + \sum_{n \geq n_2} |a_n| e^{\sigma \lambda_n} \leq \frac{1}{2} e^{\Gamma(\sigma)} + \sum_{n \geq n_2} \frac{e^{\sigma \lambda_n}}{e^{\widetilde{\Phi}(\lambda_n)}} \\ &= \frac{1}{2} e^{\Gamma(\sigma)} + e^{\Gamma(\sigma)} \sum_{n \geq n_2} \frac{e^{\sigma \lambda_n - \Gamma(\sigma)}}{e^{\widetilde{\Phi}(\lambda_n)}} \leq e^{\Gamma(\sigma)} \left(\frac{1}{2} + \sum_{n \geq n_2} \frac{e^{\widetilde{\Gamma}(\lambda_n)}}{e^{\widetilde{\Phi}(\lambda_n)}} \right) \leq e^{\Gamma(\sigma)}. \end{aligned}$$

Hence, $\sigma_a(F) \geq A$, so that $F \in \mathcal{D}_A(\lambda)$. Furthermore, $\ln M(\sigma, F) \leq \Gamma(\sigma)$, $\sigma \in [\sigma_2, A)$. □

Proof of Theorem 6. Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $T_0 > t_0 \geq 0$ be some constants. Assume that the condition (15) holds, i.e. for some $c \in (t_0, T_0)$ we have $\Delta(c, T_0) < 1$. Consider the function $y = \Delta(c, t)$, $t \in (c, +\infty)$. It follows from the properties of this function, described above, that there exists a point $T \in (c, T_0)$ such that $\Delta(c, T) < 1$. Let $q \in (\Delta(c, T), 1)$. Then there exists $n_0 \in \mathbb{N}_0$ such that

$$\ln n \le q\left(c\widetilde{\Phi}\left(\frac{\lambda_n}{c}\right) - T\widetilde{\Phi}\left(\frac{\lambda_n}{T}\right)\right), \quad n \ge n_0,$$

and thus

$$\sum_{n=0}^{\infty} \frac{1}{e^{c\widetilde{\Phi}(\lambda_n/c) - T\widetilde{\Phi}(\lambda_n/T)}} < +\infty. \tag{19}$$

Consider some Dirichlet series $F \in \mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) = t_0$. Then $t_\Phi(F) < c$, and hence $\ln \mu(\sigma, F) \le c\Phi(\sigma)$, $\sigma \in [\sigma_1, A)$. By Theorem 7, in view of (19), the series F belong to the class $\mathcal{D}_A(\lambda)$ and for this series the inequality $\ln M(\sigma, F) \le T\Phi(\sigma)$ holds for all $\sigma \in [\sigma_2, A)$, so that $T_\Phi(F) \le T < T_0$. □

Proof of Theorem 5. In view of Theorem 6, it remains only to prove the necessity of the condition (15).

We suppose that this condition is false, i. e. $\Delta(c, T_0) \ge 1$ for all $c \in (t_0, T_0)$, and prove that there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ of the form (2) such that $t_\Phi(F) = t_0$, but $T_\Phi(F) \ge T_0$.

For every $t_2 > t_1 > 0$ we set

$$\delta(t_1, t_2) = \varliminf_{n\to\infty} \frac{\ln n}{(t_2 - t_1)\Phi(\varphi(\lambda_n/t_1))}.$$

Note that $\Delta(t_1, t_2) \ge \delta(t_1, t_2)$, by the right of the inequalities (10).

First we consider the case when for every $c \in (t_0, T_0)$ the inequality $\delta(c, T_0) \ge 1$, stronger than the inequality $\Delta(c, T_0) \ge 1$, holds. By Lemma 7, for every fixed $c \in (t_0, T_0)$ and $q \in (0, 1)$, the set $E(c, q)$ of all $n \in \mathbb{N}_0$ such that simultaneously

$$\ln n \ge q(T_0 - c)\Phi\left(\varphi\left(\frac{\lambda_n}{c}\right)\right), \quad \Phi\left(\varphi\left(\frac{\lambda_{[n/2]}}{c}\right)\right) \ge q\Phi\left(\varphi\left(\frac{\lambda_n}{c}\right)\right),$$

is infinite. Let (c_k) be a decreasing to t_0 sequence of points in (t_0, T_0) and (q_k) be a increasing to 1 sequence of points in $(0, 1)$. Choose a sequence (n_k) of nonnegative integers such that for every $k \in \mathbb{N}_0$ the conditions $n_k \in E(c_k, q_k)$ and $[n_{k+1}/2] > n_k$ hold.

Let $n \in \mathbb{N}_0$. Put $b_n = e^{-c_k\widetilde{\Phi}(\lambda_n/c_k)}$, if $n \in [[n_k/2], n_k]$ for some $k \in \mathbb{N}_0$, and let $b_n = 0$, if $n \notin [[n_k/2], n_k]$ for all $k \in \mathbb{N}_0$. Consider the Dirichlet series (18) with the coefficients b_n. This series we can write as

$$G(s) = \sum_{k=0}^{\infty} \sum_{n=[n_k/2]}^{n_k} \frac{e^{s\lambda_n}}{e^{c_k\widetilde{\Phi}(\lambda_n/c_k)}}. \tag{20}$$

For all $n \in \mathbb{N}_0$ such that $n \in [[n_k/2], n_k]$ for some $k \in \mathbb{N}_0$ we obtain

$$\frac{1}{\lambda_n}\ln\frac{1}{b_n} = \frac{c_k}{\lambda_n}\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) = \overline{\Phi}\left(\frac{\lambda_n}{c_k}\right).$$

Since, by Lemma 3, the function $\overline{\Phi}$ is increasing to A on $(x_0, +\infty)$, we have $\beta(G) = A$. Thus, $G \in \mathcal{D}_A^*(\lambda)$. Furthermore, if $\psi : (A_0, A) \to (x_0, +\infty)$ be the inverse function of $\overline{\Phi}$ (here $A_0 = \overline{\Phi}(x_0 + 0)$), then for all $n \in [[n_k/2], n_k]$ and for every $k \ge k_0$ we have

$$\frac{\lambda_n}{\psi\left(\frac{1}{\lambda_n}\ln\frac{1}{b_n}\right)} = c_k.$$

This implies that $t_\Phi(G) = t_0$.

If $G \in \mathcal{D}_A(\lambda)$, then it is enough to set $a_n = b_n$ for all $n \in \mathbb{N}_0$, i. e. it is enough to set $F = G$. Indeed, if $\sigma_k = \varphi(\lambda_{n_k}/c_k)$, then for each $k \in \mathbb{N}_0$ and for all $n \in [[n_k/2], n_k]$ we have

$$\begin{aligned}\sigma_k\lambda_n - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) &= \lambda_n\varphi\left(\frac{\lambda_{n_k}}{c_k}\right) - \lambda_n\varphi\left(\frac{\lambda_n}{c_k}\right) + c_k\Phi\left(\varphi\left(\frac{\lambda_n}{c_k}\right)\right)\\ &\ge c_k\Phi\left(\varphi\left(\frac{\lambda_n}{c_k}\right)\right) \ge c_k\Phi\left(\varphi\left(\frac{\lambda_{[n_k/2]}}{c_k}\right)\right) \ge c_kq_k\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{c_k}\right)\right),\end{aligned}$$

and hence

$$\begin{aligned}M(\sigma_k, G) = G(\sigma_k) &\ge \sum_{n=[n_k/2]}^{n_k} \frac{e^{\sigma_k\lambda_n}}{e^{c_k\widetilde{\Phi}(\lambda_n/c_k)}}\\ &\ge \frac{n_k}{2}e^{c_kq_k\Phi(\varphi(\lambda_{n_k}/c_k))} \ge e^{q_k(T_0-c_k)\Phi(\varphi(\lambda_{n_k}/c_k)) - \ln 2}e^{c_kq_k\Phi(\varphi(\lambda_{n_k}/c_k))} = e^{q_kT_0\Phi(\sigma_k) - \ln 2}.\end{aligned}$$

Therefore, $\ln M(\sigma_k, G) \ge q_kT_0\Phi(\sigma_k) - \ln 2$ for each $k \in \mathbb{N}_0$. Since $\sigma_k \to A$, $k \to \infty$, we obtain

$$T_\Phi(F) = T_\Phi(G) \ge \varlimsup_{k\to\infty} \frac{\ln M(\sigma_k, G)}{\Phi(\sigma_k)} \ge T_0 \varlimsup_{k\to\infty} q_k = T_0.$$

If $G \notin \mathcal{D}_A(\lambda)$, then, by Theorem 8, there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ of the form (2) such that either $a_n = b_n$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $F(\sigma) \ge e^{T_0\Phi(\sigma)}$ for all $\sigma \in [\sigma_0, A)$. It is clear that $t_\Phi(F) = t_0$ and $T_\Phi(F) \ge T_0$.

Hence, in the case when for every $c \in (t_0, T_0)$ the inequality $\delta(c, T_0) \ge 1$ holds the existence of a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ with $t_\Phi(F) = t_0$ and $T_\Phi(F) \ge T_0$ is proved. Now let us consider the opposite case, i. e. suppose that for some $d_0 \in (t_0, T_0)$ we have $\delta(d_0, T_0) < 1$. Then

$$\ln p < (T_0 - d_0)\Phi\left(\varphi\left(\frac{\lambda_p}{d_0}\right)\right) - \ln 3, \quad p \ge p_0.$$

Since, by Lemma 3, the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$, for every $c \in (t_0, d_0]$ we obtain

$$\ln p < (T_0 - c)\Phi\left(\varphi\left(\frac{\lambda_p}{c}\right)\right) - \ln 3, \quad p \ge p_0. \tag{21}$$

By the above assumption, $\Delta(c, T_0) \ge 1$ for all $c \in (t_0, T_0)$. Then from the properties of the function $y = \Delta(t, T_0)$, $t \in (0, T_0)$, described above, it follows that for every $c \in (t_0, T_0)$ the stronger inequality $\Delta(c, T_0) > 1$ holds.

Let (c_k) be a decreasing to t_0 sequence of points in $(t_0, c_0]$. Since $\Delta(c_k, T_0) > 1$ for every $k \in \mathbb{N}_0$, there exists a sequence (n_k) of nonnegative integers such that $n_0 \ge 2p_0$ and for all $k \in \mathbb{N}_0$ we have $[n_{k+1}/2] > n_k$ and

$$\ln n_k > c_k\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{c_k}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{T_0}\right). \tag{22}$$

Let $n \in \mathbb{N}_0$. Put $b_n = e^{-c_k\widetilde{\Phi}(\lambda_n/c_k)}$, if $n \in [[n_k/2], n_k]$ for $k \in \mathbb{N}_0$, and let $b_n = 0$, if $n \notin [[n_k/2], n_k]$ for every $k \in \mathbb{N}_0$. Consider the Dirichlet series (18) with the coefficients b_n. This series we can write in the form (20). Arguing as above, we see that $\beta(G) = A$ and $t_\Phi(G) = t_0$.

Using (21) with $c = c_k$ and $p = [n_k/2]$ and also (22), for each $k \in \mathbb{N}_0$ we obtain

$$(T_0 - c_k)\Phi\left(\varphi\left(\frac{\lambda_{[n_k/2]}}{c_k}\right)\right) > \ln\left[\frac{n_k}{2}\right] + \ln 3 > \ln n_k > c_k\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{c_k}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{T_0}\right)$$
$$= \int_{c_k}^{T_0}\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{t}\right)\right)dt \ge (T_0 - c_k)\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{T_0}\right)\right)$$

and thus

$$\frac{\lambda_{[n_k/2]}}{c_k} > \frac{\lambda_{n_k}}{T_0}. \tag{23}$$

Put $\sigma_k = \varphi(\lambda_{n_k}/T_0)$. Then for every $k \in \mathbb{N}_0$ and for all $n \in [[n_k/2], n_k]$, using (22), the monotonicity of the function φ, and (23), we have

$$\sigma_k\lambda_n - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) = \lambda_n\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) - T_0\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{T_0}\right)\right) + T_0\Phi(\sigma_k)$$
$$= (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) + T_0\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{T_0}\right) + T_0\Phi(\sigma_k)$$
$$> (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) + c_k\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{c_k}\right) - \ln n_k + T_0\Phi(\sigma_k)$$
$$= (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) + c_k\int_{\lambda_n/c_k}^{\lambda_{n_k}/c_k}\varphi(x)dx - \ln n_k + T_0\Phi(\sigma_k)$$
$$\ge (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) + c_k\left(\frac{\lambda_{n_k}}{c_k} - \frac{\lambda_n}{c_k}\right)\varphi\left(\frac{\lambda_n}{c_k}\right) - \ln n_k + T_0\Phi(\sigma_k)$$
$$= (\lambda_{n_k} - \lambda_n)\left(\varphi\left(\frac{\lambda_n}{c_k}\right) - \varphi\left(\frac{\lambda_{n_k}}{T_0}\right)\right) - \ln n_k + T_0\Phi(\sigma_k)$$
$$\ge -\ln n_k + T_0\Phi(\sigma_k).$$

If $G \in \mathcal{D}_A(\lambda)$, then it is enough to set $a_n = b_n$ for all $n \in \mathbb{N}_0$, i. e. it is enough to set $F = G$. Indeed, in this case for every $k \in \mathbb{N}_0$ we obtain

$$M(\sigma_k, G) = G(\sigma_k) \ge \sum_{n=[n_k/2]}^{n_k}\frac{e^{\sigma_k\lambda_n}}{e^{c_k\widetilde{\Phi}(\lambda_n/c_k)}} \ge \frac{n_k}{2}e^{-\ln n_k + T_0\Phi(\sigma_k)} = e^{T_0\Phi(\sigma_k) - \ln 2}.$$

Hence, $\ln M(\sigma_k, G) \ge T_0\Phi(\sigma_k) - \ln 2$ for all $k \in \mathbb{N}_0$. Since $\sigma_k \to A$, $k \to \infty$, we have $T_\Phi(F) = T_\Phi(G) \ge T_0$.

If $G \notin \mathcal{D}_A(\lambda)$, then, by Theorem 8, there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ of the form (2) such that either $a_n = b_n$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $F(\sigma) \ge e^{T_0\Phi(\sigma)}$ for all $\sigma \in [\sigma_0, A)$. It is clear that $t_\Phi(F) = t_0$ and $T_\Phi(F) \ge T_0$. □

Proof of Theorem 2. Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, and $\Phi \in \Omega_A$. Suppose that the condition (11) holds and consider a Dirichlet series $F \in \mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) < +\infty$. Set $t_0 = t_\Phi(F)$. Let $T_0 > t_0$ and $c \in (t_0, T_0)$ be fixed numbers. Using the condition (11) with $t = T_0$ and left of the inequalities (10), for all $n \ge n_0$ we obtain

$$\ln n \le \frac{T_0 - c}{2}\Phi\left(\varphi\left(\frac{\lambda_n}{T_0}\right)\right) \le \frac{1}{2}\left(c\widetilde{\Phi}\left(\frac{\lambda_n}{c}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_n}{T_0}\right)\right)$$

and thus $\Delta(c, T_0) \le 1/2 < 1$. By Theorem 6, the series F belong to the class $\mathcal{D}_A(\lambda)$ and for this series the inequality $T_\Phi(F) < T_0$ holds. Since $T_0 > t_0$ is arbitrary, this inequality implies that $T_\Phi(F) = t_\Phi(F)$. □

Proof of Theorem 1. In view of Theorem 2, it remains only to prove the necessity of the condition (11). Suppose that this condition is false, i. e. there exist positive constants t_0 and δ such that

$$\varlimsup_{n\to\infty}\frac{\ln n}{\Phi(\varphi(\lambda_n/t_0)))} \ge \delta. \tag{24}$$

Set $T_0 = t_0 + \delta$. Then, using the right of the inequalities (10), for every $c \in (t_0, T_0)$ we obtain

$$c\widetilde{\Phi}\left(\frac{\lambda_n}{c}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_n}{T_0}\right) \le (T_0 - c)\Phi\left(\varphi\left(\frac{\lambda_n}{c}\right)\right) \le \delta\Phi\left(\varphi\left(\frac{\lambda_n}{t_0}\right)\right), \quad n \ge n_0.$$

Together with (24) this implies that $\Delta(c, T_0) \ge 1$ for every $c \in (t_0, T_0)$. Then, by Theorem 5, there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $t_\Phi(F) = t_0$ and $T_\Phi(F) \ge T_0 > t_0$. This completes the proof of Theorem 1. □

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Нехай Φ — така неперервна на $[\sigma_0, A)$ функція, що $\Phi(\sigma) \to +\infty$, якщо $\sigma \to A - 0$, де $A \in (-\infty, +\infty]$. Знайдено необхідну і достатню умову на невід'ємну зростаючу до $+\infty$ послідовність $(\lambda_n)_{n=0}^{\infty}$, за якої для кожного абсолютно збіжного в півплощині $\operatorname{Re} s < A$ ряду Діріхле вигляду $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s = \sigma + it$, виконується співвідношення

$$\varlimsup_{\sigma\uparrow A}\frac{\ln M(\sigma, F)}{\Phi(\sigma)} = \varlimsup_{\sigma\uparrow A}\frac{\ln \mu(\sigma, F)}{\Phi(\sigma)},$$

де $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$ і $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \ge 0\}$ — максимум модуля і максимальний член цього ряду відповідно.

Ключові слова і фрази: ряд Діріхле, максимум модуля, максимальний член, узагальнений тип.

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ZABAVSKY B.V., PIHURA O.V.

GELFAND LOCAL BEZOUT DOMAINS ARE ELEMENTARY DIVISOR RINGS

We introduce the Gelfand local rings. In the case of commutative Gelfand local Bezout domains we show that they are an elementary divisor domains.

Key words and phrases: Gelfand ring, Bezout domain, elementary divisor domain.

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INTRODUCTION

As a common generalization of local and (von Neumann) regular rings, Contessa in [1] called that a ring R is a *VNL (von Neuman local)* ring if for each $a \in R$ either a or $1-a$ is a (von Neumann) regular element. In this analogy, we consider Gelfand local rings which are generalizations of commutative domains in which each nonzero prime ideal is contained in a unique maximal ideal. In this paper we show that a commutative Gelfand local Bezout domain is an elementary divisor ring. Note that these results are responses to open questions in [6].

We introduce the necessary definitions and facts. All rings considered will be commutative and have identity. A ring is a *Bezout ring*, if every its finitely generated ideal is principal. A ring R is *an elementary divisor ring* if every matrix A (not necessarily square one) over R admits diagonal reduction, that is, there exist invertible matrices P and Q such that PAQ is a diagonal matrix, say (d_{ii}), for which d_{ii} is a divisor of $d_{i+1,i+1}$ for each i.

Two rectangular matrices A and B are *equivalent* if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$ (see [5], [6]). Recall that a ring R is called a *Gelfand ring* if for every $a, b \in R$ such that $a + b = 1$ there exist $r, s \in R$ such that $(1 + ar)(1 + bs) = 0$. A ring R is called a *PM-ring* if each prime ideal is contained in a unique maximal ideal.

RESULTS

Definition 1. *An element $a \in R$ of a commutative ring R is called a Gelfand element if the factor ring R/aR is a PM-ring.*

Proposition 1. *An element a of a commutative Bezout domain R is a Gelfand element if and only if for every elements $b, c \in R$ such that $aR + bR + cR = R$ an element a can be represented as $a = rs$, where $rR + bR = R$, $sR + cR = R$.*

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Proof. Denote $\overline{R} = R/aR$ and $\overline{b} = b + aR$, $\overline{c} = c + aR$. Since $aR + bR + cR = R$, we have $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Let $\overline{r} = r + aR$, $\overline{s} = s + aR$. Since $a = rs$, then $\overline{0} = \overline{r}\overline{s}$, where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Then $\overline{R}$ is a Gelfand ring. By [4], $\overline{R}$ is a PM-ring.

If $\overline{R}$ is a PM-ring, then $\overline{R}$ is a Gelfand ring and $\overline{0} = \overline{r}\overline{s}$, where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ for arbitrary $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Whence we obtain $aR + bR + cR = R$ and $rs \in aR$, that is, $rs = at$ for some $t \in R$.

Let $rR + aR = r_1R$, $sR + aR = s_1R$, where $r = r_1r_0$, $a = r_1a_0$, $s = s_1s_2$, $a = s_1a_2$, such that $r_0R + a_0R = R$ and $s_2R + a_2R = R$. Since $r_0R + a_0R = R$, we obtain $r_0u + a_0v = 1$ for some elements $u, v \in R$. Since $rs = at$, then $r_1r_0s = r_1a_0t$ and $r_0s = a_0$. By the equality $r_0u + a_0v = 1$ we have $a_0(tu + sv) = s$. Therefore, $a = r_1a_0$ where $r_1R + bR = R$ and $a_0R + cR = R$. □

Proposition 2. *The set of all Gelfand elements of a commutative Bezout domain R is a saturated multiplicatively closed set.*

Proof. Let a, b be Gelfand elements of R. We show that ab is a Gelfand element. Suppose the contrary. Then there exists a prime ideal P and maximal ideals M_1, M_2 of R such that $M_1 \neq M_2$ and $ab \in P \subset M_1 \cap M_2$. Since $ab \in P$ and P is a prime ideal of R, we obtain that $a \in P$ or $b \in P$. This is impossible, because a, b are Gelfand elements and $P \subset M_1 \cap M_2$. Therefore, the set of Gelfand elements is multiplicatively closed.

Let ab be a Gelfand element of R. If a is not a Gelfand element then there exists a prime ideal P such that $a \in P$ and $P \subset M_1 \cap M_2$ for some maximal ideals M_1, M_2 for which $M_1 \neq M_2$. Therefore, $ab \in P$ and $P \subset M_1 \cap M_2$, $M_1 \neq M_2$. This is impossible, because ab is a Gelfand element. □

Definition 2. *A commutative ring is a Gelfand local ring (GLR) if for each $a \in R$ either a or $1 - a$ is a Gelfand element.*

Since in a commutative domain in which each nonzero prime ideal is contained in a unique maximal ideal every nonzero element is a Gelfand element, we obtain the following result.

Proposition 3. *A commutative domain in which each nonzero prime ideal is contained in a unique maximal ideal is a Gelfand local ring.*

The following example of a Gelfand ring is due to Henriksen [2].

Let $R = \{z_0 + a_1x + a_1x^2 + \ldots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$. The Jacobson radical of R is $J(R) = \{a_1x + a_1x^2 + \ldots | a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$. Obviously, if $0 \neq a \notin J(R)$ then a is a Gelfand element. If $a \in J(R)$ then $1 - a$ is a Gelfand element.

Proposition 4. *A commutative domain is a GLR ring if and only if for every $a, b \in R$ such that $aR + bR = R$ either a or b is a Gelfand element.*

Proof. Let R be a GLR ring and $aR + bR = R$. Then $au + bv = 1$ for some elements $u, v \in R$. By the definition of R we obtain that au or $bv = 1 - au$ is a Gelfand element. If au is a Gelfand element, then by Proposition 2, a is a Gelfand element as well. If bv is a Gelfand element then by Proposition 2, b is a Gelfand element as well. Sufficiency is obvious. □

The main result of this paper is the following theorem.

Theorem 1. *Any GLR Bezout domain is an elementary divisor ring.*

Proof. Let R be a commutative GLR Bezout domain. Let $a, b, c \in R$ be such that $aR + bR + cR = R$. Let $aR + cR = dR$. Since $aR + bR + cR = R$, then $bR + dR = R$. Since R is GLR, then there two cases are possible:

1) b is a Gelfand element;

2) d is a Gelfand element.

Let us consider the first case. If b is a Gelfand element, we have $b = rs$ where $rR + aR = R$, $sR + cR = R$. Let $p \in R$ be such that $sp + ck = 1$ for some $k \in R$. Hence $rsp + rck = r$ and $bp + crk = r$. Denoting $rk = q$, we obtain $(bp + cq)R + aR = R$. Let $pR + qR = \delta R$ and $\delta = pp_1 + qq_1$ with $p_1R + q_1R = R$. Hence $p_1R + (bp_1 + cq_1)R = R$. Since $pR \subset p_1R$, we obtain $p_1R + cR = R$ and $p_1R + (bp_1 + cq_1)R = R$. Since $bp + cq = \delta(bp_1 + cq_1)$ and $(bp + cq)R + aR = R$, we obtain $(bp_1 + cq_1)R + aR = R$. Finally, we have $ap_1R + (bp_1 + cq_1)R = R$. By [3] a commutative Bezout domain R is an elementary divisor ring if and only if the matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $aR + bR + cR = R$ has a diagonal reduction. Note that a matrix A has a diagonal reduction if and only if there exist $p, q \in R$ such that $apR + (bp + cq)R = R$. That is, if b is a Gelfand element, R is an elementary divisor domain.

Consider the second case. Let d be a Gelfand element. Since $dR = aR + cR$ then $a = da_0$, $c = dc_0$, where $a_0R + c_0R = R$. Since R is a GLR ring, by Proposition 4 we obtain that an element a_0 or c_0 is a Gelfand element. Note, according to the Proposition 2 then a matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $aR + bR + cR = R$ is equivalent to the matrix B and $B = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$, where β is a Gelfand element and $\alpha R + \beta R + \gamma R = R$. By similar considerations as in case 1, we conclude that a matrix B and hence a matrix A has a diagonal reduction. Therefore R is an elementary divisor domain. □

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Забавський Б.В., Пігура О.В. *Локально гельфандові області Безу є кільцями елементарних дільників* // Карпатські матем. публ. — 2015. — Т.7, №2. — C. 188–190.

Введено локально гельфандові кільця. У випадку комутативних локально гельфандових областей Безу показано, що вони є областями елементарних дільників.

Ключові слова і фрази: гельфандове кільце, область Безу, область елементарних дільників.

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МАСЛЮЧЕНКО О.В.[1,2], ОНИПА Д.П.[2]

ГРАНИЧНІ КОЛИВАННЯ НЕПЕРЕРВНИХ ФУНКЦІЙ

У цій роботі доведено, що для довільної напівнеперервної зверху функції $f : F \to [0; +\infty]$, що визначена на межі $F = \overline{G} \setminus G$ деякої відкритої множини G в метризовному просторі X, існує неперервна функція $g : G \to \mathbb{R}$, граничне коливання $\widetilde{\omega}_g$ якої рівне f.

Ключові слова і фрази: граничне коливання, дискретно досяжний простір, напівнеперервна зверху функція.

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ВСТУП

Задача про побудову функції з даним коливанням вперше розглядалася в статті П. Костирка [5], в якій було встановлено, що для довільної напівнеперервної зверху функції $f : X \to [0; +\infty]$, що визначена на метризовному берівському просторі X без ізольованих точок, існує функція $g : X \to \mathbb{R}$, коливання якої рівне f. Ці дослідження були продовжені в роботах С. Пономарьова, Я. Еверт, З. Ґранде, З. Душинського, С. Ковальчика [4, 2, 6]. Питання про побудову функцій з певного функціонального класу з даним коливанням вивчалося в роботах [7, 9, 10, 11, 13, 14].

Ми продовжуємо дослідження функцій на межах їх областей визначення, розпочате нами в [12]. Там було встановлено, що кожна неперервна функція $f : F \to [0; +\infty)$, визначена на замкненій ніде не щільній множині $F \subseteq \mathbb{R}$ без ізольованих точок, є граничним коливанням деякої локально сталої функції $g : G \to \mathbb{R}$, що визначена на доповненні $G = \mathbb{R} \setminus F$. Досі не з'ясовано, чи можна побудувати таку локально сталу функцію g для довільної напівнеперервної зверху функції $f : F \to [0; +\infty]$. В даній роботі буде доведено існування неперервної функції g з такими властивостями, чим буде дано відповідь на проблему 1 з [12] для випадку, коли P — це властивість неперервності.

Нагадаємо, що для деякої підмножини D топологічного простору X, і деякої функції $g : D \to \mathbb{R}$, коливання цієї функції $\omega_g : \overline{D} \to [0; +\infty]$ визначається формулою

$$\omega_g(x) = \inf_{U\text{- окіл } x} \sup_{u,v \in U \cap D} |g(u) - g(v)|, \quad x \in \overline{D}.$$

Верхня та нижня граничні функції $g^\vee, g^\wedge : \overline{D} \to \overline{\mathbb{R}} = [-\infty; +\infty]$ визначаються формулами

$$g^\vee(x) = \limsup_{u \to x} g(u) = \inf_{U\text{- окіл } x} \sup_{u \in U \cap G} g(u),$$

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$$g^{\wedge}(x) = \liminf_{u\to x} g(u) = \sup_{U\text{- окіл } x} \inf_{u\in U\cap G} g(u), \quad x \in \overline{D}.$$

Як відомо, $\omega_g = g^{\vee} - g^{\wedge}$. Множина $\operatorname{supp} g = \{x \in D : g(x) \neq 0\}$ називається *носієм функції* g. *Граничним коливанням* називається звуження $\widetilde{\omega}_g = \omega_g|_{\overline{D}\setminus D}$.

1 Випадок дискретної області визначення

Нагадаємо, що ніде не щільна підмножина E топологічного простору X називається *слабко парно досяжною* [8], якщо для довільної відкритої множини G в X, такої, що $E \subseteq \overline{G} \setminus G$, існують неперетинні відкриті множини $A, B \subseteq G$, такі, що $\overline{A} \setminus G = \overline{B} \setminus G = E$. Простір X називатимемо *слабко парно досяжним*, якщо кожна замкнена ніде не щільна в X множина є слабко парно досяжною. Підмножину S метричного простору X називатимемо *ε-відокремною* [7], якщо $d(s,t) \geq \varepsilon$ для довільних різних точок $s, t \in S$. Казатимемо, що S *відокремна*, якщо вона є ε-відокремною для деякого $\varepsilon > 0$. Крім того, позначатимемо

$$B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}, \quad B(E,\varepsilon) = \bigcup_{x\in E} B(x,\varepsilon), \quad d(x,E) = \inf_{y\in E} d(x,y),$$

для $\varepsilon > 0$, $x \in X$ і $E \subseteq X$. Множину S називатимемо *σ-дискретною*, якщо існує послідовність дискретних множин S_n, така, що $S = \bigcup\limits_{n=1}^{\infty} S_n$.

Теорема 1. *Нехай X — метризовний топологічний простір, F замкнена в X і D дискретна в X, такі, що $F = \overline{D} \setminus D$ і $f : F \to [0;+\infty]$ напівнеперервна зверху. Тоді існує $g : D \to [0;+\infty)$ така, що $\widetilde{\omega}_g = f$.*

Доведення. Зафіксуємо метрику d, що породжує топологію X. З [7, лема 3] випливає, що існує функція $f_1 : F \to [0;+\infty)$, така, що $f_1^{\vee} = f$, і носій $S = \operatorname{supp} f_1$ є σ-дискретним в F. Кожна σ-дискретна підмножина метризовного простору подається у вигляді зліченного об'єднання відокремних множин [7, лема 2], зокрема, існує диз'юнктна послідовність відокремних множин S_n таких, що $S = \bigsqcup\limits_{n=1}^{\infty} S_n$.

Покажемо, що множину D можна подати у вигляді $D = D_1 \sqcup D_2$, так, що $\overline{D_1} \cap \overline{D_2} = F$. Покладемо $X_0 = \overline{D}$. Оскільки простір X_0 метризовний, то він є слабко парно досяжним [8]. Далі з того, що всі точки множини D є ізольованими, випливає, що D відкрита в X_0. Але множина $F \subseteq \overline{D} \setminus D$ є слабко парно досяжною. Тому існують неперетинні відкриті в X_0 множини $A, B \subseteq D$ такі, що $\overline{A} \setminus D = \overline{B} \setminus D = F$. Покладемо $D_1 = A$ і $D_2 = D \setminus A$. Тоді $\overline{D}_1 \setminus D = \overline{A} \setminus D = F$. Оскільки $D_2 = D \setminus A \supseteq B$, то $\overline{D}_2 \setminus D \supseteq \overline{B} \setminus D = F$. Крім того, $\overline{D}_2 \setminus D \subseteq \overline{D} \setminus D = F$. Отже, $\overline{D}_2 \setminus D = F$. Таким чином, ми довели, що $\overline{D}_1 \setminus D = \overline{D}_2 \setminus D = F$. Оскільки D — дискретний підпростір, то всі його підмножини замкнені в D. Зокрема, матимемо, що $\overline{D}_1 \cap D = D_1$ і $\overline{D}_2 \cap D = D_2$. Тепер отримуємо, що

$$\overline{D_1} \cap \overline{D_2} = ((\overline{D_1} \cap \overline{D_2}) \setminus D) \cup ((\overline{D_1} \cap \overline{D_2}) \cap D) = ((\overline{D_1} \setminus D) \cap (\overline{D_2} \setminus D))$$
$$\cup ((\overline{D_1} \cap D) \cap (\overline{D_2} \cap D)) = (F \cap F) \cup (D_1 \cap D_2) = F \cup \varnothing = F.$$

Оскільки множини S_n відокремні, то і для деякої послідовності чисел δ_n множини S_n будуть δ_n-відокремними. Виберемо деяку нескінченно малу послідовність $\varepsilon_n < \delta_n$ так, щоб $0 < \varepsilon_n < \varepsilon_{n-1}$ для довільного $n > 1$. Тоді множини S_n будуть ε_n-відокремними. Побудуємо сім'ї точок $(p_n(x) : x \in S, \quad n \in \mathbb{N})$ так, щоб для довільних $x, y \in S$, $n, m \in \mathbb{N}$ виконувались умови:

$$p_n(x) \in D_1; \tag{1}$$

$$p_n(x) \neq p_m(y), \text{ якщо } (n,x) \neq (m,y); \tag{2}$$

$$d(x, p_n(x)) < \frac{\varepsilon_{n+m}}{3}, \ x \in S_m. \tag{3}$$

Побудуємо спочатку точки $p_n(x)$ для $x \in S_1$. Зафіксуємо деяке $x \in S_1$. Тоді $x \in S_1 \subseteq S \subseteq F \subseteq \overline{D}_1$. Міркуючи індуктивно по $n \in \mathbb{N}$, виберемо точки $p_n(x) \in B(x, \frac{\varepsilon_{n+1}}{3}) \cap D_1 \setminus \{p_k(x) : k < n\}$. Припустимо, що для деякого $m > 1$ уже побудовані точки $p_n(x)$ для $n \in \mathbb{N}$, $k < m$ і $x \in S_k$ з виконанням умов (1) — (3). Оскільки для таких x матимемо, що $p_n(x) \to x$, то множина $F(x) = \{p_n(x) : n \in \mathbb{N}\} \cup \{x\}$ замкнена. Крім того, для $x \in S_k, k < m$ маємо, що $F(x) \subseteq B(x, \frac{\varepsilon_k}{3})$. З того, що S_k є ε_k-відокремними випливає, що сім'я куль $\{B(x, \frac{\varepsilon_k}{3}), x \in S_k\}$ є дискретною. А значить, дискретною буде і сім'я $\{F(x) : x \in S_k\}$. Отже, множини $F_k = \bigcup\limits_{x\in S_k} F(x)$ замкнені. Крім того, $F(x) \cap F = \{x\}$ для кожного $x \in S_k$. Тому $F_k \cap F = S_k$. Зафіксуємо точку $x \in S_m$. Визначимо послідовність точок $p_n(x)$, що задовольняють умови (1) — (3). Оскільки $x \in S_m \subseteq F \subseteq \overline{D}_1$ і $x \notin S_k = F_k \cap F$ для $k < m$, то існує $p_1(x) \in B(x, \frac{\varepsilon_{m+1}}{3}) \cap D_1 \setminus (\bigcup\limits_{k<m} F_k)$. Припустимо, що для деякого $n > 1$ вже визначені $p_j(x)$ для $j < n$. Тоді виберемо $p_n(x) \in B(x, \frac{\varepsilon_{n+m}}{3}) \cap D_1 \setminus \left(\{p_j(x) : j < n\} \cup \bigcup\limits_{k<m} F_k\right)$. Зрозуміло, що умови (1) — (3) виконуються. Таким чином, сім'я $(p_n(x) : n \in \mathbb{N}, x \in S)$ побудована.

Для довільних $x \in S$ і $E \subseteq F$ покладемо

$$P(x) = \{p_n(x) : n \in \mathbb{N}\}, \quad P(E) = \{p_n(x) : n \in \mathbb{N}, x \in E \cap S\}.$$

Доведемо, що виконується така властивість:

$(*)$ $\qquad \overline{P(E)} \cap F \subseteq E$ для довільної замкненої множини $E \subseteq F$.

Візьмемо замкнену множину $E \subseteq F$ і позначимо $E_m = E \cap S_m$. Оскільки $\bigcup\limits_{m=1}^{\infty} E_m = E \cap \bigcup\limits_{m=1}^{\infty} S_m = E \cap S$, то $P(E) = \bigcup\limits_{m=1}^{\infty} P(E_m)$. Далі зауважимо, що $P(E_m) = \bigcup\limits_{x\in E_m} P(x)$. Крім того, з монотонності (ε_m) і властивості (3) матимемо, що $P(x) \subseteq B(x, \frac{\varepsilon_m}{3})$ при $x \in E_m$, адже $d(x, p_n(x)) < \frac{\varepsilon_{n+m}}{3} < \frac{\varepsilon_m}{3}$ при $x \in E_m$. Але множина E_m є ε_m-відокремною. Тоді сім'я $(B(x, \frac{\varepsilon_m}{3}))_{x\in E_m}$, а значить і сім'я $(P(x))_{x\in E_m}$ є дискретною. Крім того, оскільки $p_n(x) \to x$, то $\overline{P(x)} \cap F = \{x\}$ для кожного $x \in E_m$. Отже,

$$\overline{P(E_m)} \cap F = \overline{\bigcup_{x\in E_m} P(x)} \cap F = \bigcup_{x\in E_m} \overline{P(x)} \cap F = \bigcup_{x\in E_m} \{x\} = E_m \subseteq E.$$

Далі позначимо $G_m = B(E, \varepsilon_m)$. Знову використавши (3), матимемо, що $P(E_k) \subseteq G_m$ при $k \geq m$. Таким чином, для довільного $m \in \mathbb{N}$ маємо, що

$$\overline{P(E)} = \overline{\bigcup_{k<m} P(E_k) \cup \bigcup_{k\geq m} P(E_k)} = \bigcup_{k<m} \overline{P(E_k)} \cup \overline{\bigcup_{k\geq m} P(E_k)} \subseteq \bigcup_{k<m} \overline{P(E_k)} \cup \overline{G}_m.$$

І нарешті, оскільки за доведеним вище $\overline{P(E_k)} \cap F \subseteq E$, то $\overline{P(E)} \cap F \subseteq \bigcup\limits_{k<m} (\overline{P(E_k)} \cap F) \cup \overline{G}_m \subseteq E \cup \overline{G}_m$ для кожного номера m. Тоді з $\bigcap\limits_{m=1}^{\infty} \overline{G}_m = E$ одержуємо

$$\overline{P(E)} \cap F \subseteq \bigcap_{m=1}^{\infty} (E \cup \overline{G}_m) = E \cup \bigcap_{m=1}^{\infty} \overline{G}_m = E.$$

Отже, властивість $(*)$ доведена.

Позначимо $P = \{p_n(x) : n \in \mathbb{N},\ x \in S\}$. Зауважимо, що $P \subseteq D_1$ і $D_2 \subseteq D \setminus P$. Визначимо функцію $g : D \to [0; +\infty)$ наступним чином:

$$g(y) = \begin{cases} 0, & \text{якщо } y \in D \setminus P, \\ f_1(x), & \text{якщо } y = p_n(x) \text{ і } f_1(x) < \infty, \\ n, & \text{якщо } y = p_n(x) \text{ і } f_1(x) = \infty. \end{cases}$$

Покажемо, що g є шуканою. Зафіксуємо $x_0 \in F$. Покажемо спершу, що $g^{\wedge}(x_0) = 0$. Візьмемо деякий окіл U точки x_0. Оскільки $\overline{D_2} \setminus D = F$, то існує $u \in D_2 \cap U$. За означенням функції g маємо, що $g(u) = 0$. Крім того, $g(x) \geq 0$ для кожного $x \in D$. Тому $\inf\limits_{u \in U} g(u) = 0$. В такому разі $g^{\wedge}(x_0) = \sup\limits_{U - \text{окіл } x_0} \inf\limits_{u \in U} g(u) = 0$.

Покажемо тепер, що $g^{\vee}(x_0) = f(x_0)$. Доведемо спочатку, що $g^{\vee}(x_0) \geq f(x_0)$. Якщо $f(x_0) = 0$, то ця нерівність очевидна. Нехай $f(x_0) > 0$. Візьмемо $\gamma \in (0; f(x_0))$ і деякий окіл U точки x_0. Оскільки $\sup f_1(U) \geq f_1^{\vee}(x_0) = f(x_0) > \gamma$, то існує $u_0 \in U$ таке, що $f_1(u_0) > \gamma$. З того, що $p_n(u_0) \to u_0$ при $n \to \infty$, випливає, що існує $n_0 \in \mathbb{N}$ таке, що для довільного $n \geq n_0$ матимемо $p_n(u_0) \in U$. За означенням функції g маємо, що $g(p_n(u_0))) \to f_1(u_0)$ при $n \to \infty$. Тому існуватиме $n_1 > n_0$ таке, що для довільного $n \geq n_1$ виконується нерівність $g(p_n(u_0)) > \gamma$. Отже, $\sup\limits_{u \in U \cap D} g(u) \geq g(p_{n_1}(u_0)) > \gamma$. Але U — довільний окіл x_0. Тому $g^{\vee}(x_0) = \inf\limits_{U - \text{окіл } x_0} \sup\limits_{u \in U} g(u) > \gamma$. Спрямувавши γ до $f(x_0)$, матимемо, що $g^{\vee}(x_0) \geq f(x_0)$.

Перевіримо, що $g^{\vee}(x_0) \leq f(x_0)$. Якщо $f(x_0) = \infty$, то все ясно. Нехай $f(x_0) < \infty$. Візьмемо $\varepsilon > 0$ і доведемо, що $g^{\vee}(x_0) \leq f(x_0) + \varepsilon$. Оскільки f напівнеперервна зверху, то існує відкритий окіл U_1 точки x_0, для якого $f(x) < f(x_0) + \varepsilon$ при $x \in U_1$. Розглянемо замкнену множину $E = F \setminus U_1$. За властивістю $(*)$ матимемо, що $\overline{P(E)} \cap F \subseteq E$. Далі, оскільки $x_0 \in F$ і $x_0 \notin E$, то $x_0 \notin \overline{P(E)}$. Тому відкрита множина $U_0 = U_1 \setminus \overline{P(E)}$ є околом точки x_0.

Покажемо, що $g(y) < f(x_0) + \varepsilon$ при $y \in U_0 \cap D$. Візьмемо $y \in U_0 \cap D$. Якщо $y \notin P$, то $g(y) = 0 < f(x_0) + \varepsilon$. Нехай $y \in P$. Тоді існують $n \in \mathbb{N}$ і $x \in S$ такі, що $y = p_n(x)$. Але $p_n(x) = y \in U_0 = U_1 \setminus \overline{P(E)}$. Тому $p_n(x) \notin P(E)$. Отже, $x \notin E$. Значить, $x \in F \setminus E = F \setminus (F \setminus U_1) = F \cap U_1 \subseteq U_1$. Отже, $g(y) \leq f_1(x) \leq f(x) \leq f(x_0) + \varepsilon$. Таким чином, $g(y) < f(x_0) + \varepsilon$ при $y \in U_0 \cap D$. Отже, $g^{\vee}(x_0) = \inf\limits_{U - \text{окіл } x_0} \sup\limits_{y \in U \cap D} g(y) \leq \sup\limits_{y \in U_0 \cap D} g(y) \leq f(x_0) + \varepsilon$. Залишилось спрямувати $\varepsilon \to 0$. □

2 ПРОДОВЖЕННЯ ЗА ДУҐУНЖІ

Для метризовного простору Y, замкненої множини A в Y і неперервної функції $f : A \to \mathbb{R}$ розглянемо покриття $\mathcal{B} = \{B(x, \frac{1}{4}d(x, A)) : x \in Y \setminus A\}$ метризовного простору $Y \setminus A$. За [3] існує локально скінченне розбиття одиниці $(\varphi_s)_{s \in S}$ на $Y \setminus A$, яке підпорядковане покриттю $\mathcal{B}$. Позначимо $U_s = \operatorname{supp}\varphi_s$, тоді $\{U_s : s \in S\}$ — локально скінченне покриття $Y \setminus A$, вписане в покриття $\mathcal{B}$. Для кожного $s \in S$ існує $x_s \in Y \setminus A$ таке, що $U_s \subseteq B(x_s, \frac{1}{4}d(x_s, A))$. За означенням відстані від точки до множини існує така точка $a_s \in A$, що $d(x_s, a_s) < \frac{5}{4}d(x_s, A)$.

Покладемо $g(x) = \triangle_{A,Y} f(x) = \begin{cases} \sum\limits_{s \in S} \varphi_s(x) f(a_s), & x \in Y \setminus A \\ f(x), & x \in A. \end{cases}$

Функція $g = \triangle_{A,Y} f(x) : Y \to [0; +\infty)$ називається *продовженням за Дуґунжі* функції $h : D \to [0; +\infty)$. В роботі [1] доведено, що g є неперервним продовженням f.

Лема 1. *Нехай X — метризовний топологічний простір, $Y \subseteq X$, A замкнена в Y, $f : A \to \mathbb{R}$ — неперервна функція і $g = \triangle_{A,Y} f : Y \to \mathbb{R}$ — продовження за Дуґунжі функції f. Тоді в кожній точці $x_0 \in \overline{A} \setminus Y$ виконується, що $\omega_f(x_0) = \omega_g(x_0)$.*

Доведення. Нехай U_s, x_s, a_s такі як в означенні функції $\triangle_{A,Y}$. Зафіксуємо точку $x_0 \in \overline{A} \setminus Y$ і покажемо, що $\omega_f(x_0) = \omega_g(x_0)$. Позначимо $S_x = \{s \in S : x \in U_s\}$, матимемо $g(x) = \sum\limits_{s \in S_x} \varphi_s(x) f(a_s)$ для $x \in Y \setminus A$.

За означенням верхньої та нижньої граничних функцій маємо, що для кожного $\varepsilon > 0$ існує такий окіл U точки x_0, що для кожного $x \in U \cap A$ виконується нерівність:

$$\alpha = f^{\wedge}(x_0) - \varepsilon < f(x) < f^{\vee}(x_0) + \varepsilon = \beta.$$

Виберемо таке $\delta > 0$, що $B(x_0, 3\delta) \subseteq U$. Позначимо $U_0 = B(x_0, \delta)$. Далі зафіксуємо деякі $x \in U_0 \cap (Y \setminus A)$ і $s \in S_x$. Тоді $x \in U_s \subseteq B(x_s, \frac{1}{4}d(x_s, A))$. Звідси $d(x, x_s) < \frac{1}{4}d(x_s, A)$. З іншого боку $d(x_s, a_s) < \frac{5}{4}d(x_s, A)$. Значить, $d(x, a_s) \leq d(x, x_s) + d(x_s, a_s) < \frac{1}{4}d(x_s, A) + \frac{5}{4}d(x_s, A) = \frac{3}{2}d(x_s, A)$. Таким чином, ми довели, що $d(x, a_s) < \frac{3}{2}d(x_s, A)$.

Візьмемо $a \in A$. Тоді $d(x, A) \leq d(x, a)$. Оскільки $x_0 \in \overline{A}$, то $d(x, A) \leq \lim\limits_{a \to x_0} d(x, a) = d(x, x_0) < \delta$, адже $x \in U_0$. Отже, $d(x, A) < \delta$. Тепер $d(x_s, A) \leq d(x_s, x) + d(x, A) < \frac{1}{4}d(x_s, A) + \delta$. Звідси $\frac{3}{4}d(x_s, A) < \delta$, а отже, $d(x_s, A) < \frac{4}{3}\delta$. Тоді $d(x, a_s) < \frac{3}{2}d(x_s, A) < \frac{3}{2} \cdot \frac{4}{3}\delta = 2\delta$. За нерівністю трикутника маємо, що $d(x_0, a_s) \leq d(x_0, x) + d(x, a_s) < \delta + 2\delta = 3\delta$.

Отже, ми довели, що для довільних $x \in U_0 \cap (Y \setminus A)$ і $s \in S_x$ виконується, що $a_s \in B(x_0, 3\delta) \subseteq U$. Тоді для функції f виконується, що $\alpha < f(a_s) < \beta$. Відповідно матимемо

$$\alpha = \sum_{s \in S_x} \varphi_s(x) \cdot \alpha < g(x) = \sum_{s \in S_x} \varphi_s(x) f(a_s) < \sum_{s \in S_x} \varphi_s(x) \cdot \beta = \beta.$$

Таким чином $\alpha < g(x) < \beta$ для кожного $x \in U_0 \cap (Y \setminus A)$. Якщо $x \in U_0 \cap A$, то з того, що $U_0 \subseteq U$, випливає, що $g(x) = f(x) \in (\alpha, \beta)$.

Отже, ми довели, що для кожного $x \in U_0 \cap Y$ виконується нерівність $\alpha < g(x) < \beta$. Значить,

$$\omega_g(x_0) \leq \omega_g(U_0) = \sup_{x', x'' \in U_0 \cap Y} |g(x') - g(x'')| \leq \beta - \alpha = f^{\vee}(x_0) + \varepsilon - (f^{\wedge}(x_0) - \varepsilon)$$
$$= f^{\vee}(x_0) - f^{\wedge}(x_0) + 2\varepsilon = \omega_f(x_0) + 2\varepsilon.$$

Спрямувавши $\varepsilon \to 0$, матимемо $\omega_g(x_0) \leq \omega_f(x_0)$. З іншого боку, функція g є продовженням функції f, а тому $\omega_g(x_0) \geq \omega_f(x_0)$. Таким чином, $\omega_g(x_0) = \omega_f(x_0)$. □

3 ОСНОВНИЙ РЕЗУЛЬТАТ

Підмножину E топологічного простору X називатимемо *слабко дискретно досяжною* [8], якщо для довільної відкритої в X множини G, такої, що $E \subseteq \overline{G} \setminus G$, існує замкнена дискретна в G множина A, така, що $\overline{A} \setminus G = E$. В [8] доведено, що в метризовному просторі усі замкнені ніде не щільні множини є слабко дискретно досяжними.

Теорема 2. *Нехай X — метризовний топологічний простір, G відкрита в X, $F = \overline{G} \setminus G$ і $f : F \to [0; +\infty]$ — напівнеперервна зверху функція. Тоді існує неперервна функція $g : G \to \mathbb{R}$ така, що $\widetilde{\omega}_g = f$.*

Доведення. За означенням дискретної досяжності існує дискретна множина $D \subseteq G$ така, що $\overline{D} \setminus D = \overline{G} \setminus G = F$. За теоремою 1 існує $h : D \to [0; +\infty)$, така, що $\widetilde{\omega}_h = f$. Підпростір D є замкненим в G. Нехай функція $g = \triangle_{D,G} h$. В [1] доведено, що g є неперервним продовженням h. За лемою 1 матимемо, що для довільного $x \in F = \overline{D} \setminus G$ виконується, що $\omega_g(x) = \omega_h(x)$. Отже, $\widetilde{\omega}_g = \widetilde{\omega}_h$. Але $\widetilde{\omega}_h = f$. Тому $\widetilde{\omega}_g = f$. Отже, функція g є шуканою. □

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Maslyuchenko O. V., Onypa D. P. *The limiting oscillations of continuous functions.* Carpathian Math. Publ. 2015, **7** (2), 191–196.

We prove that for any upper semicontinuous function $f : F \to [0; +\infty]$ defined on the boundary $F = \overline{G} \setminus G$ of some open set G in metrizable space X there is a continuous function $g : G \to \mathbb{R}$ such that its limiting oscillation $\widetilde{\omega}_g$ equals f.

Key words and phrases: limiting oscillation, discreetly attainable space, upper semicontinuous function.

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ВІДОКРЕМЛЮВАЛЬНІ ПОЛІНОМИ ТА РІВНОМІРНО АНАЛІТИЧНІ І ВІДОКРЕМЛЮВАЛЬНІ ФУНКЦІЇ

Наведено основні результати з теорії відокремлювальних поліномів та рівномірно аналітичних та відокремлювальних функцій на сепарабельних дійсних банахових просторах. Розглянуто основні властивості відокремлювальних поліномів та рівномірно аналітичних та відокремлювальних функцій. Вказано зв'язок між слабкою поліноміальною топологією та топологією норми за наявності відокремлювального полінома на просторі. Наведено достатні умови існування рівномірно аналітичних та відокремлювальних функцій. Досліджено композицію рівномірно аналітичної та відокремлювальної функції та лінійного відображення.

Ключові слова і фрази: відокремлювальні поліноми, відокремлювальні функції, аналітичні функції.

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Вступ

У 1954 році Я. Курцвейл у праці [17] встановив умови апроксимації неперервних функцій аналітичними на відкритих підмножинах сепарабельних дійсних банахових просторів. Автором, у цьому випадку, доведено що достатньою умовою для апроксимації є існування відокремлювального полінома на просторі. У 2012 у праці [2], за наявності відокремлювального полінома, встановлено зв'язок між слабкою поліноміальною топологією та топологією норми на банаховому просторі. У подальших дослідженнях апроксимації найбільш ґрунтовний результат отримали М. Боісо та П. Гаєк, у 2001 році у праці [9]. Зокрема вони довели, що для випадку апроксимації рівномірно неперервної функції на просторі умова існування відокремлювального полінома може бути послаблена до існування рівномірно аналітичної і відокремлювальної функції. Проте, незважаючи на подальші дослідження (зокрема, працю [8] авторів Д. Азгарда, Р. Фрай, Л. Кінер), таких достатніх умов існування апроксимації досі не вдалося позбутися. Тому актуальним залишається питання про те, які саме простори допускають відокремлювальні поліноми та відокремлювальні рівномірно аналітичні функції.

З теорії відокремлювальних поліномів на банахових просторах суттєві результати отримано у 1989 році М. Фабіаном, Д. Преіссом, Дж. Вайтфіелдом та В. Зізлером у статті [13] та дано ґрунтовний огляд у 1997 році Р. Гонзало, Х. Хараміло у статті [14]. З теорії рівномірно аналітичних і відокремлювальних функції на банахових просторах основні

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результати викладені у 2001 році у праці [9]. Проте всі ці результати викладено англійською мовою, крім того після 2001 року отримано деякі нові результати як з теорії відокремлювальних поліномів, так і з теорії рівномірно аналітичних і відокремлювальних функцій, зокрема праці [2, 6, 4]. Метою цієї статті є ґрунтовний огляд сучасних результатів у цих напрямках. Оскільки дана стаття є оглядовою, то значна частина результатів у ній подається без доведень які є наведеними у попередніх статтях з цієї тематики, проте надаються посилання на доведення цих результатів.

1 Відокремлювальні поліноми та їх властивості

Всі поліноми, котрі ми будемо розглядати в цій статті, вважаються неперервними.

Існує кілька (не еквівалентних) означень відокремлювального полінома. З них найбільш вживаними є наступні.

Означення 1.1. *Нехай X є нормованим простором над полем дійсних чисел $\mathbb{R}$. Дійсний поліном $q : X \to \mathbb{R}$ називається відокремлювальним поліномом, якщо q задовольняє умову*

$$\inf_{x\in X\ ||x||=1} |q(x) - q(0)| > 0. \quad (1)$$

Це означення неявно введене Курцвейлем у [18].

Означення 1.2. *Нехай X є нормованим простором над полем дійсних чисел $\mathbb{R}$. Дійсний поліном $q : X \to \mathbb{R}$ називається відокремлювальним поліномом, якщо q задовольняє умови:*

1. *$q(0) = 0$,*
2. *$|q(x)| \geq 1$ для кожного $x \in X$, такого що $||x|| = 1$.*

Це означення часто використовують у літературі.

У праці [14] введено не еквівалентне до попередніх наступне означення відокремлювального полінома.

Означення 1.3. *Нехай X є нормованим простором над полем дійсних чисел $\mathbb{R}$. Дійсний поліном $q : X \to \mathbb{R}$ називається відокремлювальним поліномом, якщо q задовольняє умови:*

1. *$q(0) = 0$,*
2. *$\inf\{q(x) : ||x|| = 1\} > 0$.*

Означення 1.4. *Нехай X є нормованим простором над полем дійсних чисел $\mathbb{R}$. Дійсний поліном $q : X \to \mathbb{R}$ називається відокремлювальним поліномом, якщо q задовольняє умови:*

1. *$q(0) = 0$,*
2. *$\inf\{|q(x)| : ||x|| = 1\} > 0$.*

Проте, як легко переконатися, питання про існування відокремлювального полінома на просторі має однакову відповідь у сенсі класичних означень 1.1, 1.2 та означення 1.3. Надалі ми будемо використовувати означення 1.3.

1.1 Властивості відокремлювальних поліномів.

Означення 1.5. *Поліном $P \in \mathcal{P}(^nX)$ називають поліномом скінченного типу, якщо він є скінченною сумою скінченних добутків лінійних функціоналів.*

Простір поліномів скінченного типу позначають $\mathcal{P}_f(^nX)$.

Означення 1.6. *Поліном $P \in \mathcal{P}(^nX)$ називають апроксимовним, якщо він належить до замикання множини всіх поліномів скінченного типу.*

Зауважимо, що всі апроксимовні поліноми скінченного типу є слабко неперервними, оскільки всі лінійні функціонали є слабко неперервними.

Зауважимо, що якщо розмірність простору X дорівнює одиниці, то кожне ненульове лінійне відображення L таке, що $L(0) = 0$, наприклад $L(x) = x$, буде відокремлювальним поліномом.

У випадку, коли розмірність нормованого простору X над полем $\mathbb{R}$ (або $\mathbb{C}$) є не меншою за два, з результатів Александрова [1] випливає, що сфера в просторі X є лінійно зв'язною множиною. У подальшому будемо вважати, що простори X та Y мають розмірність не меншу за два.

Твердження 1.1 ([5]). *Відокремлювальний поліном нескінченновимірного простору X не є апроксимовним.*

Доведення. За теоремою Голдстайна [3, ст. 460] образ одиничної сфери з X слабко щільний в одиничній сфері другого спряженого простору X''. Отже, існує напрямленість на одиничній сфері в X, яка слабко прямує до нуля. Оскільки поліном є відокремлювальним, його значення на цій напрямленості не прямують до нуля. Отже, він не є слабко неперервним. Враховуючи, що всі поліноми скінченного типу є слабко неперервними, вказаний поліном не наближається поліномами скінченного типу, а, отже, він не є апроксимовним. □

Гільбертів простір є найпростішим прикладом нескінченновимірного простору, який допускає відокремлювальний поліном. Насправді, якщо B — білінійна форма визначена скалярним добутком гільбертового простору H, тоді поліном $q(x) = B(x,x) = ||x||^2$ є відокремлювальним поліномом на H. З іншого боку, припустимо, що X є банаховим простором, що допускає однорідний відокремлювальний поліном q степеня 2. Нехай A є білінійною симетричною формою, асоційованою з q, і нехай $\alpha := \inf\{|q(x)| : ||x|| = 1\}$. З однорідності випливає, що

$$\alpha||x||^2 \leqslant |q(x)| = |A(x,x)| \leqslant ||A||||x||^2.$$

Це означає, що

$$||x|| = (|q(x)|)^{\frac{1}{2}}$$

є гільбертовою еквівалентною нормою на просторі X (інакше кажучи простір X є ізоморфним до гільбертового простору).

У просторах ℓ_{2n} та L_{2n} для $n \in \mathbb{N}$ існують відокремлювальні поліноми, які ми описуємо у наступному прикладі.

Приклад 1. *Визначимо поліном F у дійсному просторі ℓ_{2n} поклавши*

$$F(x) = \|x\|^2 = \sum_{i=1}^{\infty} x_i^{2n}, \qquad x = (x_1, \dots, x_n, \dots) \in \ell_{2n}.$$

Легко бачити, що F є $2n$-однорідним відокремлювальним поліномом.

В загальному випадку, нехай (Ω, μ) — вимірний простір з мірою μ. На дійсному просторі $L_{2n}(\Omega, \mu)$ поліном

$$F(x) = \int_{\Omega} (x(t))^{2n} d\mu, \qquad x(t) \in L_{2n}(\Omega, \mu),$$

буде відокремлювальним $2n$-однорідним поліномом.

Простір c_0 не допускає відокремлювального полінома. Це випливає з факту, доведеного у [23], що кожен поліном на просторі c_0 є слабко секвенціально неперервним. Отже, якщо q є поліномом на просторі c_0 таким, що $q(0) = 0$ та $\{e_j\}$ є відповідним базисом на c_0, тоді

$$\inf_{f \in \mathbb{N}} |q(e_j)| = 0$$

та q не може бути відокремлювальним поліномом.

Твердження 1.2 ([14]). *Якщо на просторі X існує відокремлювальний поліном q степеня m, то існує $2(m!)$-однорідний невід'ємний відокремлювальний поліном d.*

Доведення. Нехай відокремлювальний поліном q має вигляд $q = q_0 + q_1 + \dots q_m$, де q_k — k-однорідні поліноми та q_0 є константою в X. З умови 1 означення 1.3 випливає, що $q_0 = 0$. Визначимо поліном d наступним чином

$$d := (q_1)^{2(m!)} + (q_2)^{2(m!)/2} + (q_m)^{2(m!)/m}.$$

Нескладно показати, що d є $2(m!)$-однорідним відокремлювальним поліномом на просторі X. □

Скінчена сім'я поліномів $\{q_1, q_2, \dots, q_n\}$ на просторі X називається *відокремлювальною сім'єю*, якщо для кожного $x \in X$ такого, що $||x|| = 1$, ми маємо

$$\max_{i \leqslant 1 \leqslant n} \{q_i(x)\} \geqslant 1.$$

Звичайно, якщо існує відокремлювальна сім'я поліномів $\{q_1, q_2, \dots, q_n\}$ на просторі X, то цей простір допускає відокремлювальний поліном. Насправді, розглянемо поліном $q(x) = (q_1 + q_2 + \dots + q_n)^2$, який, як легко бачити, буде відокремлювальним поліномом. Невідомою залишається відповідь на питання: чи допускає простір X відокремлювальний поліном степеня, що не перевищує m, якщо на ньому існує відокремлювальна сім'я поліномів, степені яких не перевищують m?

Властивість мати відокремлювальний поліном є інваріантною відносно ізоморфізмів, тобто якщо ми маємо ізоморфізм між двома банаховими просторами, та один з цих просторів допускає відокремлювальний поліном, тоді другий простір також допускає відокремлювальний поліном. Скінчений добуток просторів, які допускають відокремлювальний поліном, також допускає відокремлювальний поліном. Також, якщо підпростір скінченої корозмірності допускає відокремлювальний поліном, то весь простір допускає відокремлювальний поліном.

З результатів доведених у праці [15] легко отримати теорему 3.1 наведену у праці [14], яку ми сформулюємо.

Теорема 1. *Нехай простір X є банаховим простором з симетричним базисом. Тоді наступні умови є еквівалентними:*

1) *простір X допускає відокремлювальний поліном;*

2) *простір X є ізоморфний до простору ℓ_{2k} для деякого цілого k.*

З означення відокремлювального полінома випливає, що дійсний поліном $q : X \to \mathbb{R}$ *не є відокремлювальним*, якщо $q(x)$ задовольняє умову:

$$\inf_{x \in X \; ||x||=1} |q(x) - q(0)| = 0. \tag{2}$$

Лема 1.1. *Якщо дійсний поліном $q(x)$ не є відокремлювальним на кулі радіуса 1 в банаховому просторі X, то він не є відокремлювальним на кулі довільного радіуса r в X.*

Твердження 1.3. *Якщо поліном p відокремлювальний на X, то він додатно або від'ємно визначений на одиничній сфері, тобто або $p(x) > 0$ для всіх таких $x \in X$, що $||x|| = 1$, або $p(x) < 0$ для всіх таких $x \in X$, що $||x|| = 1$.*

Доведення. Оскільки сфера є лінійно зв'язною множиною, а відокремлювальний поліном є неперервною функцією, то якщо би він змінював знак на сфері, то приймав би нульове значення в деякій її точці, що суперечить означенню відокремлювального полінома. Отже, p є знаковизначеним на сфері. □

З твердження 1.3 легко випливає наступне твердження, аналог якого доведений у праці [14].

Твердження 1.4. *Якщо поліном p відокремлювальний на X, то поліном p_e, складений з однорідних компонент p парних степенів, також є відокремлювальним на X.*

З твердження 1.4 випливає, що якщо p — відокремлювальний поліном на X, то $p_e \neq 0$, тобто p має ненульову парну однорідну компоненту.

Твердження 1.5. *Якщо поліном p_e відокремлювальний на X, та додатний (від'ємний) на одиничній сфері, то поліном p_{e_+} (p_{e_-}), складений з однорідних компонент p невід'ємних (недодатних) парних степенів, також є відокремлювальним на X.*

Наступний приклад, наведений у праці [4], показує, що існує банахів простір X з безумовним, але не симетричним базисом, який допускає неоднорідний відокремлювальний поліном, жодна однорідна компонента якого не є відокремлювальною. При цьому простір X не ізоморфний до ℓ_{2n} для довільного $n \in \mathbb{N}$.

Теорема 2 ([4]). *Нехай X є рівномірно опуклим дійсним сепарабельним банаховим простором із субсиметричним базисом. Нехай G є відкритою підмножиною в X. Неперервні функції на G апроксимуються аналітичними функціями рівномірно на всьому G тоді і лише тоді, коли простір X є ізоморфним до ℓ_{2n} для деякого $n \in \mathbb{N}$.*

Приклад 2. *Нехай $n, m \in \mathbb{N}$, $n > m$. В якості X візьмемо пряму суму просторів ℓ_{2n} та ℓ_{2m} з відповідними базисами e_k та g_k. Елемент цього простору має вигляд $x = \sum_{k=1}^{\infty} a_k e_k + \sum_{k=1}^{\infty} b_k g_k$, або $(a_1, b_1, a_2, b_2, \ldots)$. Можна також вважати $x = x_1 + x_2$, якщо x_1 — проекція x на ℓ_{2n}, а x_2 — проекція x на ℓ_{2m}. Відповідно, норму x визначаємо наступним чином*

$$||x|| = ||x_1||_{\ell_{2n}} + ||x_2||_{\ell_{2m}}.$$

Нехай поліном P визначається формулою

$$P(x) := \sum_{k=1}^{\infty} a_k^{2n} + \sum_{k=1}^{\infty} b_k^{2m} = ||x_1||_{\ell_{2n}}^{2n} + ||x_2||_{\ell_{2m}}^{2m} = P_1(x) + P_2(x).$$

Легко бачити, що P є відокремлювальним поліномом степеня $2n$. Кожна з його однорідних компонент P_1 та P_2 не відокремлювальні.

Для компоненти P_2 візьмемо такий елемент $x \in X$ з нормою одиниця, що він є тотожнім нулем на ℓ_{2m} і $||x||_{\ell_{2n}} = 1$. Тоді $P_2(x) = 0$, а, отже, поліном P_2 не є відокремлювальним. Для компоненти P_1 візьмемо такий $x \in X$ з нормою одиниця, що він є тотожнім нулем на ℓ_{2n} і $||x||_{\ell_{2m}} = 1$. Тоді аналогічно $P_1(x) = 0$ і поліном P_1 також не є відокремлювальним.

Базис $\{e_k, g_k\}$ є безумовним базисом в X, але не симетричним. Справді, якщо X має симетричний базис, то за теоремою 2 простір X ізоморфний до ℓ_p для деякого парного p. Оскільки X містить доповнювальну копію ℓ_{2n}, то і простір ℓ_p має містити доповнювальну копію ℓ_{2n}. Згідно з [20] це можливо тоді і лише тоді, коли $p = 2n$. З аналогічних міркувань випливає, що $p = 2m$, але це суперечить припущенню.

Зауважимо, що аналогічно в якості X можна взяти скінченну пряму суму просторів ℓ_{2n_i} де $n_k \neq n_l$ для $k \neq l$. В цьому випадку однорідні компоненти на ℓ_{2n_i} відокремлювального полінома на X також не будуть відокремлювальними поліномами на X.

Лема 1.2. *Якщо поліном p відокремлювальний на X та всі його однорідні компоненти невід'ємно визначені на X, то для довільної послідовності $\{y_n\}_{n\in\mathbb{N}}$ елементів простору X з того, що $p(y_n) \to 0$ при $n \to \infty$ випливає, що $y_n \to 0$ при $n \to \infty$.*

Доведення. Припустимо протилежне, тобто нехай існує така послідовність $\{y_n\}_{n\in\mathbb{N}}$ елементів простору X, що $p(y_n) \to 0$, але y_n не прямує до 0. Переходячи до підпослідовності, можемо вважати, що існує таке дійсне число $\varepsilon \in (0,1)$, що $||y_n|| > \varepsilon$ для всіх n. Розглянемо послідовність $\{z_n\}_{n\in\mathbb{N}}$ елементів одиничної сфери, визначену умовою $z_n = \frac{y_n}{||y_n||}$ для всіх n. Тоді

$$p(z_n) = \sum_k \frac{1}{||y_n||^k} p_k(y_n) \leqslant \frac{1}{\varepsilon^m} \sum_k p_k(y_n) = \frac{1}{\varepsilon^m} p(y_n) \to 0 \qquad \text{при} \quad n \to \infty,$$

де m — степінь полінома p, а p_k — його однорідна компонента степеня k. Це суперечить тому, що p відокремлювальний поліном, отже, наше припущення не вірне. □

Наведемо приклад відокремлювального полінома, який змінює знак в середині кулі.

Приклад 3. *Нехай $X = \ell_2$. Для $x = \sum_k e_k x_k \in \ell_2$ розглянемо поліном $p(x) = \sum_k 2x_k^4 - x_k^2$. Легко бачити, що p — відокремлювальний поліном та $p(1) = 1$. Оскільки x_k^4 прямує до нуля швидше, ніж x_k^2, то існують сфери меншого радіусу (що не перевищують $\frac{1}{\sqrt{2}}$), на яких p є нульовим та від'ємним.*

Теорема 3. *Якщо $F : X \to Y$ поліноміальний автоморфізм (поліноміальне бієктивне відображення таке, що F^{-1} — поліном та $F(0) = 0$), $p : Y \to \mathbb{R}$ відокремлювальний поліном та всі його однорідні компоненти невід'ємно визначені на Y, тоді $p(F) : X \to R$ буде відокремлювальним поліномом.*

Доведення. Відомо, що композиція поліноміальних відображень є поліноміальним відображенням, тому $p(F)$ є поліномом. Крім того, $p(F)(0) = 0$. Припустимо, що $p(F)$ не є відокремлювальним поліномом. Тоді існує така послідовність $\{x_n\}$ елементів одиничної сфери в X, що $p(F(\{x_n\})) \to 0$ при $n \to \infty$. Нехай $F(\{x_n\}) = \{y_n\}$. Оскільки p задовольняє умови леми 1.2, то $\{y_n\} \to 0$. Оскільки $F : X \to Y$ поліноміальний автоморфізм, то $F^{-1}\{y_n\} = \{x_n\} \to 0$, що суперечить вибору $\{x_n\}$. Отже, $p(F)$ — відокремлювальний поліном. □

Наступна теорема узагальнює приклад 1.

Теорема 4 ([12]). *Нехай $1 < p, q < +\infty$. Тоді наступні твердження є еквівалентними.*

1. *Простір $X = (\bigoplus_{n=1}^{\infty} \ell_q^{(n)})_{\ell_p}$ допускає відокремлювальний поліном.*
2. *Обидва p і q є парними цілими, та p є кратне q.*

Теорема 5 ([12]). *Для $1 < p, q < +\infty$ розглянемо простір $L^p(L^q)$. Тоді наступні твердження є еквівалентними.*

1. *Простір $L^p(L^q)$ допускає відокремлювальний поліном.*
2. *Обидва p і q є парними цілими, та p є кратне q.*

1.2 Слабко поліноміальна топологія і відокремлювальні поліноми.

Визначимо *слабко поліноміальну топологію* на дійсному банаховому просторі X як найслабшу топологію w_P, відносно якої всі неперервні поліноми на X зі значеннями в полі $\mathbb{R}$ є неперервними. Ця топологія породжується прообразами відкритих множин з $\mathbb{R}$ поліноміальних функціоналів на X. Базу цієї топології утворюють околи точок $x_0 \in X$, кожен з яких залежить від скінченного набору поліномів $p_1, \ldots, p_n$ і додатних чисел $\varepsilon_1, \ldots, \varepsilon_n$ та має вигляд:

$$U(x_0)_{p_1,\ldots,p_n}^{\varepsilon_1,\ldots,\varepsilon_n} = \{x \in X \colon |p_1(x) - p_1(x_0)| < \varepsilon_1, \ldots, |p_n(x) - p_n(x_0)| < \varepsilon_n\}. \tag{3}$$

Напрямленість (x_α) збігається у топологій w_P до $x_0 \in X$ тоді (і тільки тоді), коли $p(x_\alpha) \to p(x_0)$ для кожного $p \in \mathcal{P}(X)$.

Теорема 6 ([2]). *Слабко поліноміальна топологія на дійсному просторі X співпадає з топологією норми тоді і тільки тоді, коли на X існує відокремлювальний поліном.*

Зауважимо, що оскільки неперервні поліноми розділяють точки простору X, то слабко поліноміальна топологія є гаусдорфовою. Тому, за теоремою Стоуна-Вейєрштраса (див. [7, Теорема 3.2.21]), кожна w_P-неперервна функція на X наближається поліномами рівномірно на компактах у топології w_P. У випадку, коли X допускає відокремлювальний поліном, w_P-компакти є компактами в $(X, \|\cdot\|)$ і мають порожню внутрішність, якщо $\dim X = \infty$. Проте, саме в цьому випадку (теорема Курцвейля) кожна рівномірно неперервна функція на X апроксимується аналітичними функціями рівномірно на всьому просторі. Можливий інший крайній випадок, коли слабко поліноміальна топологія співпадає зі слабкою топологією на обмежених множинах. Тоді замкнена куля в $\overline{B}_X \in X$ є w_P-відносно компактним простором і теорема Стоуна-Вейєрштраса гарантує, що кожна $*$-слабко неперервна функція на $\overline{B}_{X''}$ апроксимується поліномами рівномірно на $\overline{B}_X$. Проте і в цьому випадку може трапитись, що X допускає рівномірно аналітичну відокремлювальну функцію (як наприклад $X = c_0$) і кожна рівномірно неперервна функція апроксимується аналітичними рівномірно на X [9]. З іншого боку існує багато просторів (як наприклад ℓ_p для непарних p), для яких w_P є строго сильнішою за слабку топологію на обмежених множинах і строго слабшою за топологію норми і в яких не кожна неперервна функція наближається аналітичними на X. Ці приклади показують, що апроксимація аналітичними функціями суттєво відрізняється від поліноміальної апроксимації і умови існування такої апроксимації суттєво відрізняються від умов теореми Стоуна-Вейєрштраса.

Добре відомо, що у нескінченновимірному банаховому просторі одинична сфера є щільною в одиничній кулі у слабкій топології. Наступна теорема показує, що при певних умовах w_P має таку ж властивість.

Теорема 7 ([2]). *Нехай X — нескінченновимірний дійсний банахів простір. Одинична сфера S_X є щільною в одиничній кулі $\overline{B}_X$ у слабко поліноміальній топології тоді і тільки тоді, коли X не допускає відокремлювального полінома.*

2 Рівномірно аналітичні і відокремлювальні функції

У праці [9] введені в розгляд рівномірно аналітичні і відокремлювальні функції на банахових просторах.

Означення 2.1. *Нехай X є дійсним нормованим простором. Будемо говорити, що дійсна функція d, визначена на X, є рівномірно аналітичною і відокремлювальною, якщо вона задовольняє наступні умови:*

1) *d є дійсною аналітичною функцією на X з радіусом збіжності R_{d_x} в кожній точці $x \in X$ більшим або рівним за R_d для деякого $R_d > 0$;*

2) *існує таке $\alpha \in \mathbb{R}$, що множина $\{x \in X \ : \ d(x) < \alpha\}$ є непорожньою та лежить у відкритій одиничній кулі B.*

З умови 2) випливає, що існує таке $x_0 \in X$, що $d(x_0) = \beta < \alpha$. Враховуючи аналітичність, з умови 2) випливає, що існує таке $\alpha \in \mathbb{R}$, що множина $\{x \in X \ : \ d(x) \geq \alpha\}$ не належить одиничній кулі B.

Теорема 8. *Нехай X є сепарабельним дійсним банаховим простором. На просторі X існує рівномірно аналітична і відокремлювальна функція, якщо виконується одна з наступних умов:*

1) *на просторі X існує відокремлювальний поліном;*

2) *простір X є замкненим підпростором в c_0.*

Доведення. 1) Нехай на просторі X існує відокремлювальний поліном P. Тоді на X, згідно з твердженням 1.2, існує невід'ємний однорідний відокремлювальний поліном d. Радіус збіжності $d = \infty$, отже умова 1 означення 2.1 виконується. Зафіксуємо $\alpha = 1$. Тоді для всіх $x \in X$, $|x| < 1$, $d(x)$ лежить у відкритій одиничній кулі B. Таким чином умова 2 означення 2.1 виконується. Тому d є рівномірно аналітичною і відокремлювальною функцією.

2) В [13] показано, що наступна аналітична функція

$$d((x_n)_{n\in\mathbb{N}}) := \sum_{n=1}^{\infty} (x_n)^{2n}$$

для довільного $(x_n)_{n\in\mathbb{N}} \in c_0$ задає аналітичну норму на c_0. Легко бачити, що $\|\cdot\|_\infty$-радіус збіжності d в кожній точці $x_n \in c_0$ дорівнює одиниці (наприклад, див. [21, приклад 5.5]). Також зауважимо, що d є відокремлювальною функцією, а саме:

$$0 \in \{(x_n)_{n\in\mathbb{N}} \in c_0, \ : d((x_n)_{n\in\mathbb{N}}) < 1\} \subseteq B_{c_0}.$$

Оскільки довільний підпростір в c_0 характеризується існуванням нормуючої $*$-слабко збіжної до нуля послідовності на одиничній кулі спряженого простору, то таким самим методом, як в c_0, отримаємо функцію, яка є рівномірно аналітичною і відокремлювальною. □

Зрозуміло, що умови існування рівномірно аналітичної і відокремлювальної функції успадковуються скінченими прямими сумами просторів. За відповідних обставин можна також перейти до нескінченних прямих сум. Наприклад, припустимо, що всі члени послідовності банахових просторів $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ допускають рівномірно аналітичні і відокремлювальні функції $(d_n)_{n\in\mathbb{N}}$ з радіусами збіжності $(R_n)_{n\in\mathbb{N}}$. Припустимо, що $R_d := \inf_{n\in\mathbb{N}} R_n > 0$ і що існує така послідовність додатних цілих чисел $(a_n)_{n\in\mathbb{N}}$, що

$$\sup_{n\in\mathbb{N}} \sup_{B_{X^{\mathbb{C}}}\left(\frac{R_d}{4e}\right)} |d_n^{\mathbb{C}}|^{a_n} < 1 \text{ та } \sup_{n\in\mathbb{N}} diam\left(d_n^{-1}((-\infty, 1))\right) < +\infty,$$

де $d_n^{\mathbb{C}}$ є комплексифікацією для d_n. Отже, для $X := (\oplus_{n=1}^{\infty} X_n)_{c_0}$

$$d : (x_n)_{n\in\mathbb{N}} \in \left(\bigoplus_{n=1}^{\infty} X_n\right)_{c_0} \xrightarrow{d} d((x_n)_{n\in\mathbb{N}}) := \sum_{n=1}^{\infty} d_n(x_n)^{2na_n}$$

d є рівномірно аналітичною і відокремлювальною функцією. Тому, наприклад,

$$(c_0 \oplus \oplus_{n=1}^{\infty} \ell_{2n})_{c_0}$$

допускає рівномірно аналітичну і відокремлювальну функцію.

Оскільки повна класифікація просторів, що допускають рівномірно аналітичну і відокремлювальну функцію, є доволі складним завданням, то спробуємо її отримати для окремих випадків, коли, наприклад, $c_0 \not\hookrightarrow X$ або $\ell_p \not\hookrightarrow X$ для кожного парного p.

Перший випадок приводить до просторів з відокремлювальними поліномами [10]. Дослідимо другий випадок.

Теорема 9 ([9]). *Нехай X є банаховим простором, на якому існує рівномірно аналітична і відокремлювальна функція. Припустимо, що всі скалярні поліноми на X відображають слабко збіжні до нуля послідовності в збіжні до нуля послідовності. Тоді X є ізоморфним до підпростору в c_0.*

За результатами [22, 24] простори з властивістю Дамфорда-Петіса (зокрема, всі простори неперервних на компакті K функції $C(K)$ і всі підпростори в c_0) задовольняють згадану вище умову секвенціальної неперервності поліномів.

Зауваження 2.1. *Можна показати [16], що якщо замінити припущення рівномірно аналітичної і відокремлювальної функції в теоремі 9 на існування рівномірно аналітичної і відокремлювальної функції на відкритій обмеженій підмножині в X, то звідси випливатиме, що X є сепарабельним поліедральним простором.*

Пригадаємо результат [19], який стверджує, що кожен $C(K)$ простір, який ізоморфний до підпростору в c_0, є ізоморфний до c_0.

Наслідок 2.1. *Нехай X є банаховим простором, який ізоморфний до $C(K)$ і допускає рівномірно аналітичну і відокремлювальну функцію. Тоді X є ізоморфним до c_0.*

Зауваження 2.2. *Цей наслідок можна порівняти з [11], де показано, що кожен сепарабельний поліедральний банаховий простір (наприклад $C(K)$, де K є тотально не зв'язний) допускає відокремлювальну аналітичну опуклу функцію, визначену на деякій обмеженій опуклій множині.*

Теорема 10. *Нехай X та Y дійсні банахові простори, $f : Y \to \mathbb{R}$ є рівномірно аналітичною і відокремлювальною функцією такою, що $f(0) = 0$, $g : X \to Y$ — лінійне відображення, що не зменшує норму. Тоді композиція $f \circ g : X \to \mathbb{R}$ є рівномірно аналітичною і відокремлювальною функцією.*

Доведення. Позначимо $f \circ g$ через $\widetilde{g}$. Функція $\widetilde{g}$ буде аналітичною як композиція двох аналітичних функцій. Перевіримо, що вона задовольняє умови означення 2.1.

1. Нехай x — довільна точка простору X. Для норми k-тої компоненти $\widetilde{g}_k = f_k(g)$ розкладу функції $\widetilde{g}$ в ряд в околі точки x, ми маємо оцінку

$$||f_k(g)|| \leqslant ||f_k|| ||g||^k,$$

де f_k — k-та компонента розкладу функції $\widetilde{g}$ в околі точки $g(x)$. Нехай радіус збіжності R_{f_y} в кожній точці $y \in Y$ є не меншим, ніж $R_f(0)$. Оцінимо, радіус збіжності $R_{\widetilde{g}_x}$ функції $\widetilde{g}$ в точці x :

$$0 \leqslant \frac{1}{R_{\widetilde{g}_x}} = \limsup_{k\to\infty} ||\widetilde{g}_k||^{\frac{1}{k}} = \limsup_{k\to\infty} \left(||f_k|| ||g||^k\right)^{\frac{1}{k}} = ||g|| \limsup_{k\to\infty} (||f_k||)^{\frac{1}{k}} = \frac{||g||}{R_{f_{g(x)}}}.$$

Отже,

$$R_{\widetilde{g}_x} \geqslant \frac{R_{f_{g(x)}}}{||g||} \geqslant \frac{R_f}{||g||} > 0.$$

2. Оскільки f є рівномірно аналітичною і відокремлювальною функцією, то існує таке число $\alpha \in \mathbb{R}$, що $\{y \in Y : f(y) < \alpha\} \subset B_Y$. Нехай $x \in X$ та $\widetilde{g}(x) < \alpha$. Тоді $f(g(x)) < \alpha$ і тому $g(x) \in B_Y$. Оскільки відображення g не зменшує норму, то $x \in B_X$. Крім того, оскільки g — лінійне відображення, то $g(0) = 0$ і тому $\widetilde{g}(0) = f(g(0)) = f(0) = 0$. □

Твердження 2.1. *Розглянемо лінійний простір $X = \oplus_{k=1}^{\infty} \ell_{2k}$, який є нескінченною прямою сумою просторів ℓ_{2k}. Якщо на X задати ℓ_p норму за формулою*

$$||x|| = \left(\sum_k ||x_k||^p_{\ell_{2k}}\right)^{\frac{1}{p}}, \qquad x = (x_k) \in X, \tag{4}$$

то для непарного $p > 0$ простір X з такою нормою не буде допускати ні відокремлювального полінома, ні рівномірно аналітичної і відокремлювальної функції.

Доведення. Нехай X_0 — підпростір в X, який складається з елементів вигляду $x = \sum x_k$, де $k \in \mathbb{N}$, $x_k \in \ell_{2k}$, $x_k = (a_k, 0, \ldots, 0, \ldots)$. Тоді для довільного $x \in X_0$ за умовою $||x|| = \left(\sum |a_k|^p\right)^{1/p}$. Тому X_0 є ізоморфним до ℓ_p. Оскільки на ℓ_p для непарного p не існує відокремлювального полінома, то такого полінома не існує і на X. І оскільки на ℓ_p для непарного p не існує рівномірно аналітичної і відокремлювальної функції (бо інакше, за [9, Теорема 1] норма простору ℓ_p апроксимувалась би аналітичними функціями, що не так [17, ст. 227], то такої функції не існує і на X. □

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Mytrofanov M.A. *Separating polynomials, uniform analytical and separating functions.* Carpathian Math. Publ. 2015, 7 (2), 197–208.

We present basic results of the theory of separating polynomials, uniformly analytic and separating functions on separable real Banach spaces. We consider basic properties of separating polynomials, uniformly analytic and separating functions. We indicate a relation between weak polynomial topology and norm topology of a space, provided it admits a separating polynomial. We present sufficient conditions for the existence of analytic and uniformly separating functions. We investigate a composition of an uniformly analytic and separating function and a linear mapping.

Key words and phrases: separating polynomials, separating functions, analytical functions.

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CONVERGENCE IN $L^p[0,2\pi]$-METRIC OF LOGARITHMIC DERIVATIVE AND ANGULAR v-DENSITY FOR ZEROS OF ENTIRE FUNCTION OF SLOWLY GROWTH

The subclass of a zero order entire function f is pointed out for which the existence of angular v-density for zeros of entire function of zero order is equivalent to convergence in $L^p[0,2\pi]$-metric of its logarithmic derivative.

Key words and phrases: logarithmic derivative, entire function, angular density, Fourier coefficients, slowly increasing function.

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Introduction

Let L be the class of all positive non-decreasing unbounded continuously differentiable on $[0,+\infty)$ functions v such that $rv'(r)/v(r) \to 0$ as $0 < r_0 \le r \to +\infty$. It is known (see [1, p. 15]) that the class L coincides with the class of slowly increasing functions accurate to the equivalent functions. By $H_0(v)$, $v \in L$, we denote the class of entire functions f of zero order for which $0 < \Delta = \overline{\lim\limits_{r\to+\infty}}\, n(r)/v(r) < +\infty$. Without loss of generality we assume that $f(0) = 1$.

We will say that zeros of function $f \in H_0(v)$, $v \in L$, have an angular v- density, if the limit

$$\Delta(\alpha,\beta) = \lim_{r\to+\infty} \frac{n(r,\alpha,\beta)}{v(r)}$$

exists for all α and β, that do not belong to some no more than countable set from $[0,2\pi]$. Here $n(r,\alpha,\beta)$ is the number of zeros a_n of the function f, which lie in the sector $\{z\colon |z| \le r, \alpha \le \arg z < \beta\}$, $0 \le \alpha < \beta < 2\pi$.

We also denote by $F(z) = z\dfrac{f'(z)}{f(z)}$ the logarithmic derivative of f, by $\mathcal{E}_\eta$ the family of all measurable sets $G \subset \mathbb{R}_+$ such that $\overline{\lim\limits_{r\to+\infty}}\, \mathrm{mes}(G\cap[0,r])/r \le \eta$, $0<\eta<1$.

Theorem ([2]). *Let $v \in L$, $f \in H_0(v)$ and zeros of the function f have angular v-density. Then there exists a set $G \in \mathcal{E}_\eta$ such that, for arbitrary $p \in [1,+\infty)$,*

$$\left\| \frac{F(re^{i\theta}) - n(r)}{v(r)} \right\|_p \to 0, \quad r\to+\infty, \quad r \notin G.$$

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The converse statement is false. The question is under which conditions for $f \in H_0(v)$ from the convergence in $L^p[0,2\pi]$-metric of the function F the existence of angular v-density of zeros of f will follow. We note [3], that in the case of an entire function f of non integer order $\rho > 0$ the existence of angular density of its zeros is equivalent to the following

$$\left\| \frac{F(re^{i\theta})}{r^{\rho(r)}} - g(\theta) \right\|_p \to 0,\ r \to +\infty,\ r \notin G, G \in \mathcal{E}_\eta,$$

where $p \in [1,+\infty)$, $g \in L^1[0,2\pi]$, $\rho(r)$ is the proximate order of f, $\rho(r) \to \rho$, $r \to +\infty$.

In this paper we will point out the subclass of entire function f from the class $H_0(v)$, for which the existence of angular v-density of zeros of the function f will be equivalent to the convergence of the logarithmic derivative F in $L^p[0,2\pi]$-metric.

1 Main results

Let us denote by $\Gamma_m = \bigcup\limits_{j=1}^{m} \{z\colon \arg z = \theta_j\} = \bigcup\limits_{j=1}^{m} l_{\theta_j}$, $-\pi \le \theta_1 < \theta_2 < \ldots < \theta_m < \pi$, the finite system of rays, by $n(r,\theta_j; f) = n(r,\theta_j)$ the number of zeros of $f \in H_0(v)$ lying on the ray $l_{\theta_j} = \{z\colon \arg z = \theta_j\}$ and modules of which do not exceed r. Let $h_j(\theta) = (\theta - \pi - \theta_j)$, $\theta_j < \theta < \theta_j + 2\pi$, and $\widehat{h}_j(\theta)$ be its periodic continuation from $(\theta_j, \theta_j + 2\pi)$ on $\mathbb{R}$, $j = \overline{1,m}$. For $\widetilde{v} \in L$ we set

$$v(r) = \int_0^r \frac{\widetilde{v}(t)}{t} dt.$$

It is easy to see that $v \in L$ and $\widetilde{v}(r) = o(v(r))$ as $r \to +\infty$.

Theorem 1. *Let $\widetilde{v} \in L$, $f \in H_0(v)$. Suppose that zeros of the function f lie on the finite system of rays Γ_m and for each $j = \overline{1,m}$, $\Delta_j > 0$*

$$n(r,\theta_j) = \Delta_j v(r) + o(\widetilde{v}(r)), \quad r \to +\infty. \tag{1}$$

Then

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p = \left\| \frac{F(re^{i\theta}) - \Delta v(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p \to 0,\ r \to +\infty, \tag{2}$$

where $H_f(\theta) = \sum\limits_{j=1}^{m} \Delta_j \widehat{h}_j(\theta)$, $\Delta = \sum\limits_{j=1}^{m} \Delta_j$.

Theorem 2. *Let $G \in L^1[0,2\pi]$, $\widetilde{v} \in L$, $f \in H_0(v)$. Suppose that zeros of the function f lie on the finite system of rays Γ_m and*

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iG(\theta) \right\|_p \to 0,\ r \to +\infty. \tag{3}$$

Then zeros of the function f have an angular v-density, moreover $\int\limits_0^{2\pi} G(\theta)d\theta = 0$.

2 Additional results

To prove Theorems 1, 2 we will use the following results, which we formulate as lemmas.

Lemma 1 ([1]). *Let $v \in L$. Then for $k \in \mathbb{N}$*

$$r^k \int_r^{+\infty} \frac{v(t)}{t^{k+1}} dt = \frac{1}{k} v(r) + o(v(r)), \quad r \to +\infty,$$

$$r^{-k} \int_0^r \frac{v(t)}{t^{-k+1}} dt = \frac{1}{k} v(r) + o(v(r)), \quad r \to +\infty.$$

Lemma 2. *Let $v \in L$, $\varepsilon(t)$ be a function, locally integrable on $[1,+\infty)$, and $\varepsilon(t) \to 0$ as $t \to +\infty$. Then for $k \in \mathbb{N}$*

$$r^k \int_r^{+\infty} \frac{\varepsilon(t)v(t)}{t^{k+1}} dt = o(v(r)), \quad r \to +\infty,$$

$$r^{-k} \int_0^r \frac{\varepsilon(t)v(t)}{t^{-k+1}} dt = o(v(r)), \quad r \to +\infty.$$

The proof of this lemma follows from applying L'Hopital's rule.

Let $c_k(r,\Phi)$, $k \in \mathbb{Z}$, be the Fourier coefficients of function $\Phi(re^{i\theta})$ as a function of θ, that is $c_k(r,\Phi) = \frac{1}{2\pi} \int\limits_0^{2\pi} \Phi(re^{i\theta}) e^{-ik\theta}$, $\quad r > 0$.

Lemma 3. *Let $\widetilde{v} \in L$, $f \in H_0(v)$, zeros of the function f lie on the finite system of rays Γ_m and (1) holds. Then there exists $r_0 > 0$ such that for $k \in \mathbb{Z} \setminus \{0\}$ the relations*

$$c_k(r,F) = -\frac{\widetilde{\Delta}_k}{k} \widetilde{v}(r) + o(\widetilde{v}(r)), \quad r \to +\infty,$$

$$|c_k(r,F)| \le \frac{2\Delta}{|k|} \widetilde{v}(r), \quad r \ge r_0, \Delta > 0, \widetilde{\Delta}_k > 0,$$

hold.

Proof. Since $n_k(r) = \sum\limits_{j=1}^{m} e^{-ik\theta_j} n(r,\theta_j)$, owing to (1) we have

$$n_k(r) = \widetilde{\Delta}_k v(r) + o(\widetilde{v}(r)), \quad r \to +\infty,$$

where $\widetilde{\Delta}_k = \sum\limits_{j=1}^{m} \Delta_j e^{-ik\theta_j}$.

From formulas for calculating the coefficients $c_k(r,F)$ [2, Lemma 3] and the last identity, using Lemma 2, we obtain

$$c_k(r,F) = n_k(r) - kr^k \int_r^{+\infty} \frac{n_k(t)}{t^{k+1}} dt = \widetilde{\Delta}_k v(r) + o(\widetilde{v}(r)) - k\widetilde{\Delta}_k r^k \int_r^{+\infty} \frac{v(t)}{t^{k+1}} dt - kr^k \int_r^{+\infty} \frac{o(\widetilde{v}(r))}{t^{k+1}} dt$$

$$= \widetilde{\Delta}_k v(r) - k\widetilde{\Delta}_k r^k \left(\frac{v(r)}{kr^k} + \frac{1}{k} \int_r^{+\infty} \frac{\widetilde{v}(t)}{t^{k+1}} dt \right) + o(\widetilde{v}(r)) = -\widetilde{\Delta}_k r^k \int_r^{+\infty} \frac{\widetilde{v}(t)}{t^{k+1}} dt + o(\widetilde{v}(r)),\ k \in \mathbb{N},$$

as $r \to +\infty$.

Similarly, for $k \in \mathbb{Z}$, $k < 0$,

$$c_k(r,F) = \widetilde{\Delta}_k r^k \int_0^r \frac{\widetilde{v}(t)}{t^{k+1}} dt + o(\widetilde{v}(r)), \quad r \to +\infty.$$

From this and Lemma 1 we have

$$c_k(r,F) \sim -\frac{\widetilde{\Delta}_k}{k}\widetilde{v}(r), \quad r \to +\infty,$$
$$|c_k(r,F)| \le \frac{2\Delta}{|k|}\widetilde{v}(r), \quad r \ge r_0.$$

□

3 Proof of the main results

Proof of Theorem 1. We set

$$b_k := c_k(H_f) = \frac{1}{2\pi}\sum_{j=1}^{m} \Delta_j \int_0^{2\pi} \widehat{h}_j(\theta) e^{-ik\theta} d\theta = \frac{1}{2\pi}\sum_{j=1}^{m} \Delta_j \int_{\theta_j}^{\theta_j+2\pi} h_j(\theta) e^{-ik\theta} d\theta$$
$$= \frac{i}{k}\sum_{j=1}^{m} \Delta_j e^{-ik\theta_j} = \begin{cases} \dfrac{i\widetilde{\Delta}_k}{k}, & k \ne 0, \\ 0, & k = 0. \end{cases} \tag{4}$$

Therefore $|b_k| \le \frac{\Delta}{|k|}$, $k \ne 0$. Since, by Lemma 3, $|c_k(r,F)| \le \frac{2\Delta}{|k|}\widetilde{v}(r)$, the sequence $\left(\frac{c_k(r,F)}{\widetilde{v}(r)} - ib_k\right)_{k\ne 0}$ belongs to the space l_q with $q > 1$, $r \ge r_0$. We have

$$c_k(r, F(z) - n(r)) = c_k(r,F) \qquad \text{for} \quad k \ne 0.$$

Thus by Hausdorff-Young theorem [4, p. 153] for $p \ge 2$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p \le \left\{ \sum_{k\ne 0} \left| \frac{c_k(r,F)}{\widetilde{v}(r)} - ib_k \right|^q \right\}^{\frac{1}{q}}.$$

Since the resulting series is uniformly convergent for all $r \ge r_0$, by making the limiting transition as $r \to +\infty$ in the last inequality and owing to Lemma 3 and identity (4) we obtain

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p \to 0, \ r \to +\infty,$$

for $p \ge 2$. By Holder's inequality $\|\cdot\|_p \le \|\cdot\|_2$ for $1 \le p < 2$, that is (2) is also valid for $1 \le p < 2$. The Theorem 1 is proved. □

Proof of Theorem 2. Let us denote by g_k the Fourier coefficients of function G, namely $g_k = c_k(G)$. Then, by (3), we obtain

$$\left| \frac{c_k(r,F) - n(r)}{\widetilde{v}(r)} - ig_k \right| = \left| \frac{c_k(r, F - n(r))}{\widetilde{v}(r)} - ig_k \right| = \left| \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iG(\theta) \right) e^{-ik\theta} d\theta \right|$$
$$\le \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iG(\theta) \right| d\theta \le \left\| \frac{F(re^{i\varphi}) - n(r)}{\widetilde{v}(r)} - iG(\theta) \right\|_p \to 0,$$

as $r \to +\infty$. Since $c_0(r,F) = n(r)$ from the last relation we find $g_0 = 0$, that is

$$\int_0^{2\pi} G(\theta) d\theta = 0.$$

For $k \ne 0$ owing to $l_k(r) := c_k(r, \ln f) = \int_0^r \frac{c_k(t,F)}{t} dt$ we obtain

$$ig_k = \lim_{r\to+\infty} \frac{c_k(r,F)}{\widetilde{v}(r)} = \lim_{r\to+\infty} \frac{\int_0^r c_k(t,F)/t dt}{\int_0^r \widetilde{v}(t)/t dt} = \lim_{r\to+\infty} \frac{c_k(r,\ln f)}{v(r)}. \tag{5}$$

By identities (see, for instance, [5, Lemma 1])

$$l_k(r) = -r^k \int_r^{+\infty} \frac{n_k(t)}{t^{k+1}} dt = -r^k \sum_{j=1}^{m} e^{-ik\theta_j} \int_r^{+\infty} \frac{n(r,\theta_j)}{t^{k+1}} dt, \ k = \overline{1,m},$$

we have the linear system of equations with respect to the quantities $n(r,\theta_j)$, $j = \overline{1,m}$,

$$\begin{cases} \sum_{j=1}^{m} e^{-i\theta_j} n(r,\theta_j) = rl_1'(r) - l_1(r), \\ \sum_{j=1}^{m} e^{-i2\theta_j} n(r,\theta_j) = rl_2'(r) - 2l_2(r), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{j=1}^{m} e^{-im\theta_j} n(r,\theta_j) = rl_m'(r) - ml_m(r). \end{cases}$$

Since

$$\begin{vmatrix} e^{-i\theta_1} & e^{-i\theta_2} & \dots & e^{-i\theta_m} \\ e^{-i2\theta_1} & e^{-i2\theta_2} & \dots & e^{-i2\theta_m} \\ \cdot & \cdot & \cdot & \cdot \\ e^{-im\theta_1} & e^{-im\theta_2} & \dots & e^{-im\theta_m} \end{vmatrix} \ne 0,$$

we have

$$n(r,\theta_j) = \sum_{k=1}^{m} b_{kj}(kl_k(r) - rl_k'(r)) = \sum_{k=1}^{m} b_{kj}(kl_k(r) - c_k(r,F)),$$

where $b_{kj} \in \mathbb{C}$. Taking into consideration (5) and the last identities we obtain for $j = \overline{1,m}$

$$n(r,\theta_j) = (1 + o(1)) i \sum_{k=1}^{m} b_{kj}(kg_k v(r) - g_k \widetilde{v}(r)) = i \sum_{k=1}^{m} b_{kj} k g_k v(r) + o(v(r))$$
$$= \Delta_j v(r) + o(v(r)), \quad r \to +\infty.$$

Hence, zeros of the function f have an angular v-density. □

Remark. *By the conditions of Theorem 2 it is easy to verify that* $G(\theta) = H_f(\theta)$ *for almost all* $\theta \in [0, 2\pi]$.

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Виділено підклас цілих функцій f нульового порядку, для яких поняття існування кутової v-щільності нулів f та збіжність в $L^p[0, 2\pi]$-метриці її логарифмічної похідної є рівносильними.

Ключові слова і фрази: логарифмічна похідна, ціла функція, кутова щільність, коефіцієнти Фур'є, повільно зростаюча функція.

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EL OUADIH S., DAHER R.

ON ESTIMATES FOR THE JACOBI TRANSFORM IN THE SPACE $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$

For the Jacobi transform in the space $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ we prove the estimates in some classes of functions, characterized by a generalized modulus of continuity.

Key words and phrases: Jacobi operator, Jacobi transform, Jacobi generalized translation, generalized modulus of continuity.

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1 INTRODUCTION AND PRELIMINARIES

The main aim of this paper is to generalize the Theorem 1 in [3].

Let $\alpha > \frac{-1}{2}, \alpha \geq \beta \geq \frac{-1}{2}$ and $J^{\alpha,\beta}(x) := (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$ for $x \in \mathbb{R}^+$. We define $L^p_{(\alpha,\beta)}(\mathbb{R}^+) := L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$, $1 < p \leq 2$, as the Banach space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{p,(\alpha,\beta)} = \left(\int_0^{+\infty} |f(x)|^p J^{\alpha,\beta}(x)dx\right)^{\frac{1}{p}}.$$

Let

$$D_{\alpha,\beta} := \frac{d^2}{dx^2} + ((2\alpha+1)\cos x + (2\beta+1)\,\mathrm{tg}\, x)\frac{d}{dx}$$

be the Jacobi differential operator and denote by $\varphi_\lambda^{(\alpha,\beta)}(x)$, $\lambda \in \mathbb{C}, x \in \mathbb{R}^+$, the Jacobi function of order (α, β). The function $\varphi_\lambda^{(\alpha,\beta)}(x)$ satisfies the differential equation

$$(D_{\alpha,\beta} + \lambda^2 + \rho^2)\varphi_\lambda^{(\alpha,\beta)}(x) = 0,$$

where $\rho = \alpha + \beta + 1$.

Lemma 1.1. *Let* $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}, \rho = \alpha + \beta + 1$, *and let* $x_0 > 0$. *Then for* $|\eta| \leq \rho$ *there exists a positive constant* $C_1 = C_1(\alpha, \beta, x_0)$ *such that*

$$|1 - \varphi_{\mu+i\eta}^{(\alpha,\beta)}(x)| \geq C_1|1 - j_\alpha(\mu x)|,$$

for all $0 \leq x \leq x_0$ *and* $\mu \in \mathbb{R}$, *where* $j_\alpha(x)$ *is a normalized Bessel function of the first kind.*

Proof. (See [2], Lemma 9). □

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In $L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ consider the Jacobi generalized translation T_h

$$T_h f(x) = \int_0^{+\infty} f(z)\mathcal{K}_{\alpha,\beta}(x,h,z)J^{\alpha,\beta}(z)dz,$$

where the kernel $\mathcal{K}_{\alpha,\beta}$ is explicitly known (see [5]).

The Jacobi transform is defined by formula

$$\widehat{f}(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda^{(\alpha,\beta)}(x)J^{\alpha,\beta}(x)dx.$$

The inversion formula is

$$f(x) = \frac{1}{2\pi}\int_0^{+\infty} \widehat{f}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda),$$

where $d\mu(\lambda) :=| C(\lambda) |^{-2} d\lambda$ and the C-function $C(\lambda)$ is defined by

$$C(\lambda) = \frac{2^\rho\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

We have the Young inequality

$$\|\widehat{f}\|_{q,(\alpha,\beta)} \leq K\|f\|_{p,(\alpha,\beta)}, \tag{1}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and K is positive constant.

We note the important property of the Jacobi transform: if $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$, then

$$\widehat{D_{\alpha,\beta}f}(\lambda) = -(\lambda^2+\rho^2)\widehat{f}(\lambda). \tag{2}$$

The following relation connects the Jacobi generalized translation and the Jacobi transform:

$$\widehat{T_h f}(\lambda) = \varphi_\lambda^{(\alpha,\beta)}(h)\widehat{f}(\lambda). \tag{3}$$

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = T_h f(x) - f(x) = (T_h - I)f(x),$$

where I is the identity operator in $L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1}f(x)) = (T_h-1)^k f(x) = \sum_{i=0}^{k}(-1)^{k-1}\binom{k}{i}T_h^i f(x), \tag{4}$$

where $T_h^0 f(x) = f(x)$, $T_h^i f(x) = T_h(T_h^{i-1}f(x))$, $i=1,2,\dots,k$ and $k=1,2,\dots$.

The k-th order generalized modulus of continuity of a function $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ is defined by

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta}\|\Delta_h^k f\|_{p,(\alpha,\beta)}, \qquad \delta>0.$$

Let $W^{r,k}_{p,\varphi}(D_{\alpha,\beta})$ denote the class of functions $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ that have generalized derivatives in the sense of Levi (see [4]) satisfying the estimate

$$\Omega_k(D^r_{\alpha,\beta}f,\delta) = O(\varphi(\delta^k)), \qquad \delta \to 0;$$

i.e.,

$$W^{r,k}_{p,\varphi}(D_{\alpha,\beta}) := \{f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+) : D^r_{\alpha,\beta}f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+) \text{ and } \Omega_k(D^r_{\alpha,\beta}f,\delta) = O(\varphi(\delta^k)),\ \delta\to 0\},$$

where $\varphi(x)$ is any nonnegative function given on $[0,\infty)$, and $D^0_{\alpha,\beta}f = f$, $D^r_{\alpha,\beta}f = D_{\alpha,\beta}(D^{r-1}_{\alpha,\beta}f)$; $r=1,2,\dots$

2 Main Results

In this section we estimate the integral

$$\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

in certain classes of functions in $L^p_{(\alpha,\beta)}(\mathbb{R}^+)$.

Lemma 2.1. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, $\rho = \alpha+\beta+1$, and let $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$. Then*

$$\left(\int_0^\infty (\lambda^2+\rho^2)^{qr}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{q}} \leq K\|\Delta_h^k D^r_{\alpha,\beta}f(x)\|_{p,(\alpha,\beta)},$$

where $1<p\leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From formula (2) we obtain

$$\widehat{D^r_{\alpha,\beta}f}(\lambda) = (-1)^r(\lambda^2+\rho^2)^r\widehat{f}(\lambda); \quad r=0,1,\dots \tag{5}$$

We use the formulas (3) and (5) and conclude

$$\widehat{T_h^i D^r_{\alpha,\beta}f}(\lambda) = (-1)^r(\varphi_\lambda^{(\alpha,\beta)}(h))^i(\lambda^2+\rho^2)^r\widehat{f}(\lambda), \qquad 1\leq i\leq k. \tag{6}$$

From the definition of finite difference (4) and formula (6) the image $\Delta_h^k D^r_{\alpha,\beta}f(x)$ under the Jacobi transform has the form

$$\Delta_h^k \widehat{D^r_{\alpha,\beta}f}(\lambda) = (-1)^r(\varphi_\lambda^{(\alpha,\beta)}(h)-1)^k(\lambda^2+\rho^2)^r\widehat{f}(\lambda).$$

Now by the inequality (1) we have the result. □

Theorem 1. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, $\rho=\alpha+\beta+1$ and let $f \in W^{r,k}_{p,\varphi}(D_{\alpha,\beta})$. Then*

$$\sup_{W^{r,k}_{p,\varphi}(D_{\alpha,\beta})}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{q}} = O\left(N^{-2r}\varphi\left(\left(\frac{c}{N}\right)^k\right)\right) \qquad as \quad N\to\infty,$$

where $r=0,1,2,\dots$; $k=1,2,\dots$, $c>0$ is a fixed constant, and $\varphi(t)$ is any nonnegative function defined on the interval $[0,\infty)$.

Proof. In the terms of $j_\alpha(x)$, for the normalized Bessel function of the first kind we have (see [1])

$$1-j_\alpha(x) = O(1), \quad x\geq 1, \tag{7}$$

$$1-j_\alpha(x) = O(x^2), \quad 0\leq x\leq 1, \tag{8}$$

$$\sqrt{hx}J_\alpha(hx) = O(1), \quad hx\geq 0, \tag{9}$$

where $J_\alpha(x)$ is Bessel function of the first kind, and

$$j_\alpha(x) = \frac{2^\alpha\Gamma(\alpha+1)}{x^\alpha}J_\alpha(x). \tag{10}$$

Let $f \in W^{r,k}_{p,\varphi}(D_{\alpha,\beta})$. By the Hölder inequality and Lemma 1.1, we have

$$\begin{aligned}
&\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) - \int_N^\infty j_\alpha(\lambda h)|\widehat{f}(\lambda)|^q d\mu(\lambda) = \int_N^\infty (1-j_\alpha(\lambda h))|\widehat{f}(\lambda)|^q d\mu(\lambda)\\
&= \int_N^\infty (1-j_\alpha(\lambda h))\left(|\widehat{f}(\lambda)|\mid C(\lambda)\mid^{\frac{-2}{q}}\right)^q d\lambda\\
&= \int_N^\infty (1-j_\alpha(\lambda h))\left(|\widehat{f}(\lambda)|\mid C(\lambda)\mid^{\frac{-2}{q}}\right)^{q-\frac{1}{k}}\left(|\widehat{f}(\lambda)|\mid C(\lambda)\mid^{\frac{-2}{q}}\right)^{\frac{1}{k}} d\lambda\\
&\le \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty |1-j_\alpha(\lambda h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}\\
&\le \frac{1}{C_1}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty |1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}\\
&\le \frac{1}{C_1}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty (\lambda^2+\rho^2)^{-rq+rq}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}\\
&\le \frac{(N^2+\rho^2)^{\frac{-r}{k}}}{C_1}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty (\lambda^2+\rho^2)^{qr}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}.
\end{aligned}$$

In view of Lemma 2.1, we conclude that

$$\int_N^\infty (\lambda^2+\rho^2)^{qr}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda) \le K^q\|\Delta_h^k D^r_{\alpha,\beta}f(x)\|^q_{p,(\alpha,\beta)}.$$

Therefore

$$\begin{aligned}
\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) \;\le\; &\int_N^\infty j_\alpha(\lambda h)|\widehat{f}(\lambda)|^q d\mu(\lambda)\\
&+ K^{\frac{1}{k}}\frac{(N^2+\rho^2)^{\frac{-r}{k}}}{C_1}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\|\Delta_h^k D^r_{\alpha,\beta}f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}.
\end{aligned}$$

From formulas (9) and (10), we have $j_\alpha(\lambda h) = O((\lambda h)^{-\alpha-\frac{1}{2}})$. Then

$$\begin{aligned}
&\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\\
&= O\left(\int_N^\infty (\lambda h)^{-\alpha-\frac{1}{2}}|\widehat{f}(\lambda)|^q d\mu(\lambda) + N^{-\frac{2r}{k}}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\|\Delta_h^k D^r_{\alpha,\beta}f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}\right)\\
&= O\left((Nh)^{-\alpha-\frac{1}{2}}\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) + N^{-\frac{2r}{k}}\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\|\Delta_h^k D^r_{\alpha,\beta}f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}\right),
\end{aligned}$$

or

$$(1-(Nh)^{-\alpha-\frac{1}{2}})\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-\frac{2r}{k}})\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\|\Delta_h^k D^r_{\alpha,\beta}f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}.$$

Choose a constant c such that the number $1-c^{-\alpha-\frac{1}{2}}$ is positive. Setting $h = c/N$ in the last inequality, we obtain

$$\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-\frac{2r}{k}})\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}}\varphi^{\frac{1}{k}}\left(\left(\frac{c}{N}\right)^k\right).$$

Then

$$\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(N^{-2rq}\varphi^q\left(\left(\frac{c}{N}\right)^k\right)\right),$$

which completes the proof. □

Corollary 2.1. *Let $\alpha \ge \beta \ge \frac{-1}{2}$, $\alpha \ne \frac{-1}{2}$, $\rho = \alpha+\beta+1$, $\varphi(t) = t^\nu$, $\nu > 0$, and let $f \in W^{r,k}_{p,t^\alpha}(D_{\alpha,\beta})$. Then*

$$\left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{q}} = O(N^{-2r-k\nu}) \quad as \quad N\to\infty,$$

where $1 < p \le 2$ and q such that $\frac{1}{p}+\frac{1}{q} = 1$.

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Ель Оуадіх С., Дахер Р. *Оцінки для перетворення Якобі в просторі $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$* // Карпатські матем. публ. — 2015. — Т.7, №2. — C. 215–219.

Для перетворення Якобі в просторі $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ доведено оцінки в деяких класах функцій, що характеризуються узагальненим модулем неперервності.

Ключові слова і фрази: оператор Якобі, перетворення Якобі, узагальнений зсув Якобі, узагальнений модуль неперервності.

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POPOVYCH R.

ON THE MULTIPLICATIVE ORDER OF ELEMENTS IN WIEDEMANN'S TOWERS OF FINITE FIELDS

We consider recursive binary finite field extensions $E_{i+1} = E_i(x_{i+1})$, $i \geq -1$, defined by D. Wiedemann. The main object of the paper is to give some proper divisors of the Fermat numbers N_i that are not equal to the multiplicative order $O(x_i)$.

Key words and phrases: finite field, multiplicative order, Wiedemann's tower.

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INTRODUCTION

High order elements are often needed in several applications that use finite fields [8, 9]. Ideally we want to have a possibility to obtain a primitive element for any finite field. However, if we have no the factorization of the order of finite field multiplicative group, it is not known how to reach the goal. That is why one considers less ambitious question: to find an element with provable high order. It is sufficient in this case to obtain a lower bound on the order. The problem is considered both for general and for special finite fields. We use F_q to denote finite field with q elements. Gao [5] gave an algorithm constructing high order elements for many (conjecturally all) general extensions F_{q^n} of finite field F_q with lower bound on the order $\exp(\Omega((\log m)^2/\log\log m))$. Voloch [13] proposed a method which constructs an element of order at least $\exp((\log m)^2)$ in finite fields from elliptic curves.

For special finite fields, it is possible to construct elements which can be proved to have much higher orders. Extensions connected with a notion of Gauss period are considered in [1, 11]. The lower bound on the order equals to $\exp(\Omega(\sqrt{m}))$. Extensions based on Kummer polynomials are of the form $F_q[x]/(x^m - a)$ [2, 3]. It is shown in [3] how to construct high order elements in such extensions with the condition $q \equiv 1(mod\ m)$. The lower bound $\exp(\Omega(m))$ is obtained in this case. The condition $q \equiv 1(mod\ m)$ for extensions based on Kummer polynomials is removed in [12].

Another less ambitious, but supposedly more important question, is to find primitive elements for a class of special finite fields. A polynomial algorithm that finds a primitive element in finite field of small characteristic is described in [6]. However, the algorithm relies on two unproved assumptions and is not supported by any computational example. Our paper can be considered as a step towards this direction. We give some restrictions and as a consequence

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a lower bound on multiplicative order of some elements in binary recursive extensions of finite fields defined by Wiedemann [14]. The paper concerns with the open question posed by Wiedemann [10, problem 28]. Voloch [13] gave the first nontrivial estimate for the order of elements in this construction, namely $\exp(2^{2^i\delta})$, where δ is an absolute constant. However, the constant is unknown. Our bound does not depend on any unknown constant.

More precisely, we consider the following finite fields defined by Wiedemann that are constructed recursively:

$$x_{-1} = 1,\ E_{-1} = F_2(x_{-1}) = F_2,$$

for $i \geq -1$, $E_{i+1} = E_i(x_{i+1})$, where x_{i+1} satisfies the equation

$$x_{i+1}^2 + x_{i+1}x_i + 1 = 0. \tag{1}$$

So, we obtain the following tower of characteristic two finite fields:

$$F_2 \subset E_0 = F_2(x_0) \subset E_1 = E_0(x_1) \subset \dots$$

For comparison, the following finite fields are defined by Conway [14]:

$$c_{-1} = 1,\ L_{-1} = F_2(c_{-1}) = F_2,$$

for $i \geq -1$, $L_{i+1} = L_i(c_{i+1})$, where c_{i+1} satisfies the equation

$$c_{i+1}^2 + c_{i+1} + \prod_{j=-1}^{i} c_j = 0.$$

In this case, the following tower of finite fields of characteristic two arises:

$$L_{-1} = F_2(c_{-1}) = F_2 \subset L_0 = F_2(c_0) \subset L_1 = L_0(c_1) \subset \dots$$

From a point of view of applications such construction is very attractive, since we can perform operations with finite field elements recursively, and therefore effectively [7].

Note that the number of elements of the multiplicative group E_i^* ($i \geq 0$), that is the set of non-zero elements of the field E_i, equals to $2^{2^{i+1}} - 1$. If to denote the Fermat numbers $N_j = 2^{2^j} + 1$ ($j \geq 0$), then the cardinality of E_i^* ($i \geq 0$) is equal to $2^{2^{i+1}} - 1 = \prod_{j=0}^{i} N_j$. For example, $|E_0^*| = 2^{2^1} - 1 = 3$, $|E_1^*| = 2^{2^2} - 1 = 15 = 3 \cdot 5$, $|E_2^*| = 2^{2^3} - 1 = 255 = 3 \cdot 5 \cdot 17$.

1 PRELIMINARIES

We give below in Lemmas 1–9 auxiliary results for this paper.

Lemma 1 ([5]). *For $j \geq 1$ the following equality holds $N_j = \prod_{k=0}^{j-1} N_k + 2$.*

As a consequence of Lemma 1, we have the following lemma.

Lemma 2. *Numbers N_j ($j \geq 0$) are pair-wise coprime.*

Lemma 3 ([14]). *For $i \geq 0$, the following equality holds: $(x_i)^{N_i} = 1$.*

The multiplicative order of a field element x_i is defined to be the smallest nonnegative integer N_i such that $(x_i)^{N_i} = 1$. According to Lagrange theorem for finite groups, the above result implies that the order of x_i divides N_i. In the case where N_i is prime, x_i has order that precisely equals to N_i. The open question posed by Wiedemann [10, problem 28] is as follows: does the multiplicative order $O(x_i)$ of the element x_i equal to N_i. In any case, the order of x_i divides N_i.

Lemma 4. *Let $u_r = \prod_{i=0}^{r} x_i$ for $r = 0, 1 \dots$. The multiplicative order of element u_r equals to $O(u_r) = \prod_{i=0}^{r} O(x_i)$.*

Proof. Since the Fermat numbers are pair-wise coprime (see Lemma 2), the order of $u_r = \prod_{i=0}^{r} x_i$ is the product of the orders of x_i, $0 \le i \le r$. The number of elements of the multiplicative group E_i^* $(i = 0, 1, \dots)$ is equal to $\prod_{j=0}^{i} N_j$. As a corollary of Lemma 3 we have that the group E_i^* $(i = 0, 1, \dots)$ is an internal direct product of subgroups with $N_j (j = 0, \dots, i)$ elements. The element x_i belongs to the subgroup with the order N_i. □

We say that an element of a finite field is primitive if its order is the same as the number of nonzero field elements. If the order of x_i is, in fact, N_i for $0 \le i \le r$, then $u_r = \prod_{i=0}^{r} x_i$ is a primitive element in E_r, because $2^{2^{i+1}} - 1 = \prod_{j=0}^{i} N_j$. So, the given before Wiedemann's question can be reformulated as follows: is the element $u_r = \prod_{i=0}^{r} x_i$ primitive.

Lemma 5. *For $j \ge 2$, a divisor $\alpha > 1$ of the number N_j is of the form $\alpha = l \cdot 2^{j+2} + 1$, where l is a positive integer.*

Proof. The result obtained by Euler and Lucas (see [4, Theorem 1.3.5]) states: for $j \ge 2$, a prime divisor of the number N_j is of the form $l \cdot 2^{j+2} + 1$, where l is a positive integer. Clearly a product of two numbers of the specified form is a number of the same form. Hence, the result follows. □

Lemma 6. *Let K be a finite field of characteristic two and $x, y \in K$. If*

$$y^2 = yx + 1, \tag{2}$$

then

$$y^{2^k} = yx^{2^k - 1} + \sum_{j=1}^{k} x^{2^k - 2^j} \tag{3}$$

for any positive integer k.

Proof. By induction on k. For $k = 1$ we obtain the equality (2).

Suppose the equality (2) holds for some positive integer k. Then

$$y^{2^{k+1}} = \left(y^{2^k}\right)^2 = \left(yx^{2^k-1} + \sum_{j=1}^{k} x^{2^k-2^j}\right)^2 = y^2 x^{2^{k+1}-2} + \sum_{j=1}^{k} x^{2^{k+1}-2^{j+1}}.$$

Taking into account (2), we have

$$y^{2^{k+1}} = yx^{2^{k+1}-1} + \sum_{j=1}^{k+1} x^{2^{k+1}-2^j},$$

that is the equality (3) is true for $k + 1$ as well. □

Lemma 7. *The multiplicative order $O(x_i) = N_i$ for $0 \le i \le 11$.*

Proof. For $0 \le i \le 4$ Fermat numbers are prime [4]: $N_0 = 3$, $N_1 = 5$, $N_2 = 17$, $N_3 = 257$, $N_4 = 65537$. Therefore clearly for these numbers, as a consequence of Lemma 3, the order of the element x_i coincides with the correspondent Fermat number, that is $O(x_i) = N_i$.

The rest of the proof uses computer calculations. We perform calculations of order of the element x_i for $5 \le i \le 11$. In this case Fermat numbers are completely factored into primes [5].

Using the mentioned factorizations, we calculate x_i in the power N_i/q for any prime divisor q of the number N_i. Really, if an element in the power N_i/q is not equal to one, then the element in the power of any divisor of N_i/q is also not equal to one. As a result we obtain that for $5 \le i \le 11$ the order of element x_i is not less than N_i, namely precisely equals to N_i. □

Lemma 8. *For $i \ge 0$ the inverse element to the element x_i equals to $(x_i)^{-1} = x_i + x_{i-1}$.*

Proof. Based on the given in the introduction recursive equation (1), that defines the Wiedemann's tower, we have $x_i(x_i + x_{i-1}) = (x_i)^2 + x_i x_{i-1} = 1$. Hence, the element x_i is the inverse to the element $x_i + x_{i-1}$. □

Lemma 9. *The following equalities hold for $i \ge 1$:*

$$x_i^2 = x_i x_{i-1} + 1, \tag{4}$$

$$x_i^3 = x_{i-1}(x_{i-2}x_i + 1), \tag{5}$$

$$x_i^5 = x_{i-1}[(x_{i-2}^2 + 1)x_{i-1}x_i + x_{i-2}x_{i-1} + 1]. \tag{6}$$

Proof. The equality (4) follows directly from (1). Using (4) for x_i^2 consequently two times, we obtain

$$x_i^3 = x_i^2 \cdot x_i = x_{i-1}x_i^2 + x_i = x_{i-1}^2 x_i + x_{i-1} + x_i.$$

Substituting now the value of x_{i-1}^2 from (4), leads to (5). Using (4) and (5), we have

$$\begin{aligned} x_i^5 = x_i^3 \cdot x_i^2 = x_{i-1}(x_{i-2}x_i + 1)(x_{i-1}x_i + 1) &= x_{i-1}(x_{i-2}x_{i-1}x_i^2 + x_{i-2}x_i + x_{i-1}x_i + 1) \\ &= x_{i-1}(x_{i-1}^2 x_{i-2}x_i + x_{i-2}x_{i-1} + x_{i-2}x_i + x_{i-1}x_i + 1). \end{aligned}$$

Substituting now the value of x_{i-1}^2 from (4), gives (6). □

2 MAIN RESULTS

We give in this section in Theorems 1–3 and Corollary main results of this paper.

Theorem 1. *The order $O(x_i)$ $(i \ge 0)$ cannot be a divisor of a number of the form $2^k + 1$, where k is a positive integer and $k < 2^i$.*

Proof. By induction on i. For $0 \le i \le 11$ it is true according to Lemma 7. Let the assertion holds for numbers from 12 to $i - 1$.

Show by the way of contradiction that the assertion holds for i as welll. Assume that $O(x_i)$ divides $2^k + 1$, where $k < 2^i$. Then $(x_i)^{2^k+1} = 1$ and Lemma 8 gives

$$(x_i)^{2^k} = (x_i)^{-1} = x_i + x_{i-1}. \tag{7}$$

On the other hand, putting in (3) $y = x_i$, $x = x_{i-1}$, we have

$$(x_i)^{2^k} = x_i(x_{i-1})^{2^k-1} + \sum_{j=1}^{k} (x_{i-1})^{2^k-2^j}. \tag{8}$$

Comparing coefficients near x_i in (7) and (8), we obtain $(x_{i-1})^{2^k-1} = 1$. Hence, $O(x_{i-1})$ divides $2^k - 1$. At the same time, by Lemma 3, $O(x_{i-1})$ is a divisor of $2^{2^{i-1}} + 1$. Then $O(x_{i-1})$ divides the sum of numbers $2^{2^{i-1}} + 1$ and $2^k - 1$, that is equal to $S = 2^{2^{i-1}} + 2^k$. Consider the following three possible cases.

1) If $k = 2^{i-1}$, then $S = 2^{2^{i-1}} + 2^k = 2^{2^{i-1}+1}$. In this case $O(x_{i-1})$ equals to a power of two. This contradicts to the fact that $O(x_{i-1})$ must divide $2^{2^{i-1}} + 1$.

2) If $k < 2^{i-1}$, then $S = 2^k(2^{2^{i-1}-k} + 1)$. As 2^k is coprime with $2^{2^{i-1}} + 1$, the order $O(x_{i-1})$ divides $2^{2^{i-1}-k} + 1$. Since $k \geq 1$, the inequality $2^{i-1} - k < 2^{i-1}$ holds, a contradiction with the induction hypothesis.

3) If $k > 2^{i-1}$, then $S = 2^{2^{i-1}}(2^{k-2^{i-1}} + 1)$. As $2^{2^{i-1}}$ is coprime with $2^{k-2^{i-1}} + 1$, the order $O(x_{i-1})$ is a divisor of $2^{k-2^{i-1}} + 1$. Since $k < 2^i$, the inequality $k - 2^{i-1} < 2^{i-1}$ is true, a contradiction with the induction hypothesis.

Therefore, we obtain a contradiction in all three possible cases, what shows that the assertion also holds for i. □

Theorem 2. *The order $O(x_i)$ $(i \geq 0)$ cannot be a divisor of a number of the form $s \cdot 2^k + 1$, where $s = 3, 5$ and k is a non negative integer.*

Proof. By the way of contradiction. If $O(x_i)$ is a divisor of a number of the form $s \cdot 2^k + 1$, then $(x_i)^{s \cdot 2^k + 1} = 1$ and clearly

$$(x_i)^{s \cdot 2^k} = (x_i)^{-1}. \tag{9}$$

Denote $t = 2^i - k$. Then $2^{2^i} = 2^t \cdot 2^k$. Powering left and right side of the equation (9) to 2^t and taking into account $(x_i)^{2^{2^i}} = (x_i)^{-1}$, we obtain

$$(x_i)^{2^t} = (x_i)^s.$$

Consider the case $s = 3$. According to Lemma 6

$$(x_i)^{2^t} = x_i(x_{i-1})^{2^t - 1} + \sum_{j=1}^{t} (x_{i-1})^{2^t - 2^j}. \tag{10}$$

Comparing coefficients near x_i on the right side of (10) and (5), we have

$$(x_{i-1})^{2^t - 2} = x_{i-2}.$$

Since $x_{i-2} \neq 1$ and, by lemma 2, Fermat numbers are coprime, we have the trivial intersection of cyclic subgroups $\langle x_{i-1} \rangle \bigcap \langle x_{i-2} \rangle = 1$, a contradiction. As a consequence, $O(x_i)$ $(i \geq 0)$ cannot be a divisor of a number of the form $3 \cdot 2^k + 1$, where k is a non negative integer.

Consider now the case $s = 5$. Comparing coefficients near x_i on the right side of (10) and (6), we obtain

$$(x_{i-1})^{2^t - 3} = (x_{i-2})^2 + 1.$$

Since $(x_{i-2})^2 + 1 = x_{i-2}x_{i-3} \neq 0$, we have $(x_{i-2})^2 + 1 \in [F_2(x_{i-2})]^*$. Note that $(x_{i-2})^2 + 1 \neq 1$, because $(x_{i-2})^2 \neq 0$. The fact: N_{i-1} is coprime with $N_{i-2}N_{i-3}$ (see lemma 2), leads to $\langle x_{i-1} \rangle \bigcap [F_2(x_{i-2})]^* \neq 1$, a contradiction. Therefore, $O(x_i)$ $(i \geq 0)$ cannot be a divisor of a number of the form $5 \cdot 2^k + 1$, where k is a non negative integer. □

Theorem 3. *The order of element x_i equals to N_i for $0 \leq i \leq 11$ and is at least $7 \cdot 2^{i+2} + 1$ for $i \geq 12$.*

Proof. By Lemma 7 $O(x_i) = N_i$ holds for $0 \leq i \leq 11$. Show now that $O(x_i) \geq 7 \cdot 2^{i+2} + 1$ for $i \geq 12$. If $(x_i)^{n_i} = 1$, then, by the Lagrange theorem for finite groups, n_i divides N_i. According to Lemma 3, $n_i = s \cdot 2^{i+2} + 1$, where s is a positive integer. By Theorem 1, s can not equal to 1, 2 or 4, and by Theorem 2 s can not equal to 3, 5 or 6, that is $s \geq 7$. Hence, the result follows. □

Corollary. *The order of element $u_r = \prod_{i=0}^{r} x_i$ equals to $\prod_{i=0}^{r} N_i$ for $0 \leq r \leq 11$ and is at least $\prod_{i=0}^{11} N_i \cdot \prod_{i=12}^{r} (7 \cdot 2^{i+2} + 1)$ for $r \geq 12$.*

Proof. According to Lemma 4, we have the equality $O(u_r) = \prod_{i=0}^{r} O(x_i)$. Applying now Theorem 3, we obtain given in the formulation of the corollary bounds on the order. □

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Попович Р. *Про мультиплікативний порядок елементів у вежах Відемана скінченних полів* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 220–225.

Розглядаються рекурсивні двійкові розширення скінченних полів $E_{i+1} = E_i(x_{i+1})$, $i \geq -1$, визначені Д. Відеманом. Основна мета роботи — описати деякі власні дільники чисел Ферма N_i, які не дорівнюють мультиплікативному порядку $O(x_i)$.

Ключові слова і фрази: скінченне поле, мультиплікативний порядок, вежа Відемана.

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GEOMETRY OF HYPERSURFACES OF A QUARTER SYMMETRIC NON METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD

The purpose of the paper is to study the notion of CR-submanifold and the existence of some structures on a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold. We study the existence of a Kahler structure on M and the existence of a globally metric frame f-structure in sence of Goldberg S.I., Yano K. [6]. We discuss the integrability of distributions on M and geometry of their leaves. We have tries to relate this result with those before obtained by Goldberg V., Rosca R. devoted to Sasakian manifold and conformal connections.

Key words and phrases: CR-submanifold, quasi-Sasakian manifold, quarter symmetric non metric connection, integrability conditions of the distributions.

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INTRODUCTION

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are respectively given by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [5] S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\varphi X - u(X)\varphi Y,$$

where u is a 1-form and φ is a tensor field of type $(1,1)$. Some properties of quarter symmetric connections are studied in [7]. In [8, 9] S. Rahman studied Transversal hypersurfaces of almost hyperbolic contact manifolds with a quarter symmetric non metric connections respectively.

The concept of CR-submanifold of a Kahlerian manifold has been defined by A. Bejancu [3]. Later A. Bejancu, N. Papaghiue [4] introduced and studied the notion of semi-invariant submanifold of a Sasakian manifold. These submanifolds are closely related to CR-submanifolds in a Kahlerian manifold. However the existence of the structure vector field implies some important changes.

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The paper is organized as follows. In the first section we recall some results and formulae for the later use. In the second section we prove the existence of a Kahler structure on M and the existence of a globally metric frame f-structure in sence of S.I. Goldberg, S.I. Yano. The third section is concerned with integrability of distributions on M and geometry of their leaves. In section 4 the study of conformal connections with respect to the quarter symmetric non metric connection in a quasi-Sasakian manifold is considered.

1 PRELIMINARIES

Let $\bar{M}$ be a real $2n+1$ dimensional differentiable manifold, endowed with an almost contact metric structure (f,ξ,η,g). Then we have

$$\begin{aligned}&(a)\ f^2 = -I + \eta\otimes\xi, \quad (b)\ \eta(\xi) = 1, \quad (c)\ \eta\circ f = 0, \quad (d)\ f(\xi) = 0,\\ &(e)\ \eta(X) = g(X,\xi), \quad (f)\ g(fX, fY) = g(X,Y) - \eta(X)\eta(Y)\end{aligned} \tag{1}$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. Throughout the paper, all manifolds and maps are differentiable of class C^∞. We denote by $F(\bar{M})$ the algebra of differentiable functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of sections of a vector bundle E over $\bar{M}$.

The Niyembuis tensor field, denoted by N_f, with respect to the tensor field f, is given by

$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[fX,Y] + f[X,fY]$$

for all $X,Y \in \Gamma(T\bar{M})$ and the fundamental 2-form Φ is given by $\Phi(X,Y) = g(X,fY)$ for all $X,Y \in \Gamma(T\bar{M})$. The curvature tensor field of $\bar{M}$, denoted by $\bar{R}$ with respect to the Levi-Civita connection $\bar{\nabla}$, is defined by $\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$ for all $X,Y,Z \in \Gamma(T\bar{M})$.

Definition 1. *(a) An almost contact metric manifold $\bar{M}$ (f,ξ,η,g) is called normal if*

$$N_f(X,Y) + 2d\eta(X,Y)\xi = 0 \qquad \text{for all} \quad X,Y \in \Gamma(T\bar{M}),$$

or equivalently ([1]) $(\bar{\nabla}_{fX}f)Y = f(\bar{\nabla}_X f)Y - g((\bar{\nabla}_X\xi, Y)$ for all $X,Y \in \Gamma(T\bar{M})$.
(b) The normal almost contact metric manifold $\bar{M}$ is called cosympletic if $d\Phi = d\eta = 0$.

Let $\bar{M}$ be an almost contact metric manifold $\bar{M}$. According to [1] we say that $\bar{M}$ is a quasi-Sasakian manifold if and only if ξ is a Killing vector field and

$$(\bar{\nabla}_X f)Y = g(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi \qquad \text{for all} \quad X,Y \in \Gamma(T\bar{M}). \tag{2}$$

Next we define a tensor field F of type $(1,1)$ by $FX = -\bar{\nabla}_X\xi$ for all $X \in \Gamma(T\bar{M})$.

Lemma 1. *Let $\bar{M}$ be a quasi-Sasakian manifold. Then for all $X,Y \in \Gamma(T\bar{M})$ we have*

$$\begin{aligned}&(a)\ (\bar{\nabla}_\xi f)X = 0, \quad (b)\ f\circ F = F\circ f, \quad (c)\ g(FX,Y) + g(X,FY) = 0,\\ &(d)\ F\xi = 0, \quad (e)\ \eta\circ F = 0, \quad (f)\ (\bar{\nabla}_X F)Y = \bar{R}(\xi,X)Y.\end{aligned} \tag{3}$$

The tensor field f defined on $\bar{M}$ is an f-structure in sense of Yano that is $f^3 + f = 0$.

Definition 2. *The quasi-Sasakian manifold $\bar{M}$ is said to be of rank $2p+1$ iff*

$$\eta\wedge(d\eta)^p \neq 0 \quad \text{and} \quad (d\eta)^{p+1} = 0.$$

On other hand, a quarter symmetric non metric connection $\bar{\nabla}$ on M is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X. \tag{4}$$

Using (4) in (2), we have

$$(\bar{\nabla}_X f)Y = g(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi + \eta(Y)X - \eta(X)\eta(Y)\xi, \tag{5}$$

$$\bar{\nabla}_X \xi = -FX + fX. \tag{6}$$

Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced Levi-Civita connection on M and by $TM^{\perp}$ the normal vector bundle to M. The Gauss and Weingarten formulas of hypersurfaces of a quarter symmetric non metric connections are

$$(a)\ \bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad (b)\ \bar{\nabla}_X N = -AX, \tag{7}$$

where A is the shape operator with respect to the section N. It is known that for all $X, Y \in \Gamma(TM)$

$$B(X,Y) = g(AX,Y). \tag{8}$$

Because the position of the structure vector field with respect to M is very important we prove the following result.

Theorem 1. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. If the structure vector field ξ is normal to M then $\bar{M}$ is cosympletic manifold and M is totally geodesic immersed in $\bar{M}$.*

Proof. Because $\bar{M}$ is quasi-Sasakian manifold, then it is normal and $d\Phi = 0$ ([2]). By direct calculation using (7) (b), we infer for all $X, Y \in \Gamma(T\bar{M})$

$$\begin{aligned} d\eta(X,Y) &= \frac{1}{2}\{(\bar{\nabla}_X\eta)(Y) - (\bar{\nabla}_Y\eta)(X)\} = \frac{1}{2}\{g(\bar{\nabla}_X\xi, Y) - g(\bar{\nabla}_Y\xi, X)\}, \\ 2d\eta(X,Y) &= g(AY,X) - g(AX,Y) = 0. \end{aligned} \tag{9}$$

From (7) (b) and (9) we deduce for all $X, Y \in \Gamma(T\bar{M})$

$$\begin{aligned} 0 = d\eta(X,Y) &= \frac{1}{2}\{(\bar{\nabla}_X\eta)(Y) - (\bar{\nabla}_Y\eta)(X)\} \\ &= \frac{1}{2}\{g(\bar{\nabla}_X\xi, Y) - g(\bar{\nabla}_Y\xi, X)\} = g(Y, \bar{\nabla}_X\xi) = -g(AX,Y) = 0, \end{aligned} \tag{10}$$

which proves that M is totally geodesic. From (10) we obtain $\bar{\nabla}_X\xi = 0$ for all $X \in \Gamma(T\bar{M})$. By using (6), (3) (b) and (1) (d) from the above relation we state for all $X \in \Gamma(T\bar{M})$

$$-f(\bar{\nabla}_{fX}\xi) + fX = \bar{\nabla}_X\xi, \tag{11}$$

because $fX \in \Gamma(T\bar{M})$ for all $X \in \Gamma(T\bar{M})$. Using (11) and the fact that ξ is a not Killing vector field, we deduce $d\eta \neq 0$.

Next we consider only the hypersurface which are tangent to ξ. Denote by $U = fN$ and from (1) (f), we deduce $g(U,U) = 1$. Moreover, it is easy to see that $U \in \Gamma(TM)$. Denote by $D^{\perp} = Span(U)$ the 1-dimensional distribution generated by U, and by D the orthogonal complement of $D^{\perp} \oplus (\xi)$ in TM. It is easy to see that

$$fD = D, \quad D^{\perp} \subseteq TM^{\perp}, \quad TM = D \oplus D^{\perp} \oplus (\xi), \tag{12}$$

where $\oplus$ denote the orthogonal direct sum. According with [1] from (12) we deduce that M is a CR-submanifold of $\bar{M}$. □

Definition 3. *A CR-submanifold M of a quasi-Sasakian manifold $\bar{M}$ is called CR-product if both distributions $D \oplus (\xi)$ and $D^{\perp}$ are integrable and their leaves are totally geodesic submanifold of M.*

Denote by P the projection morphism of TM to D and using the decomposion in (10) we deduce for all $X \in \Gamma(T\bar{M})$ that

$$X = PX + a(X)U + \eta(X)\xi, \qquad fX = fPX + a(X)fU + \eta(fX)\xi,$$

therefore $fX = fPX - a(X)fU$. Since

$$U = fN, \qquad fU = f^2N = -N + \eta(N)\xi = -N + g(N,\xi)\xi = -N,$$

where a is a 1-form on M defined by $a(X) = g(X,U)$, $X \in \Gamma(TM)$. From (12) using (1) (a) we infer for all $X \in \Gamma(TM)$

$$fX = tX - a(X)N, \tag{13}$$

where t is a tensor field defined by $tX = fPX$, $X \in \Gamma(TM)$. It is easy to see that

$$(a)\ t\xi = 0, \quad (b)\ tU = 0. \tag{14}$$

2 INDUCED STRUCTURES ON A HYPERSURFACE OF A QUARTER SYMMETRIC NON METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD

The purpose of this section is to study the existence of some induced structure on a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold. Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. From (1) (a), (13) and (14) we obtain $t^3 + t = 0$, that is the tensor field t defines an f-structure on M in sense of Yano [10]. Moreover, from (1) (a), (13), (14) we infer for all $X \in \Gamma(TM)$

$$t^2X = -X + a(X)U + \eta(X)\xi. \tag{15}$$

Lemma 2. *On a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$ the tensor field t satisfies for all $X, Y \in \Gamma(TM)$*

$$(a)\ g(tX,tY) = g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y), \quad (b)\ g(tX,Y) + g(X,tY) = 0. \tag{16}$$

Proof. From (1) (f), and (13) we deduce for all $X, Y \in \Gamma(TM)$

$$\begin{aligned} g(X,Y) - \eta(X)\eta(Y) &= g(fX,fY) = g(tX - a(X)N, tY - a(Y)N) \\ &= g(tX,tY) - a(Y)g(tX,N) - a(X)g(N,tY) \\ &\quad + a(X)a(Y)g(N,N) = g(tX,tY) + a(X)a(Y), \\ g(tX,tY) &= g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y), \\ g(tX,Y) + g(X,tY) &= g(fX + a(X)N, Y) + g(X, fY + a(Y)N) \\ &= g(fX,Y) + a(X)g(N,Y) + g(X,fY) + a(Y)g(X,N) \\ &= g(fX,Y) + g(X,fY) = 0. \end{aligned}$$

□

Lemma 3. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have*

$$(a)\ FU = fA\xi, \qquad (b)\ FN = A\xi, \qquad (c)\ [U,\xi] = 0. \tag{17}$$

Proof. We take $X = U$ and $Y = \xi$ in (2) $f(\bar{\nabla}_U\xi) = -\bar{\nabla}_N\xi - U$. Then using (1) (a), (6), (7) (b), we deduce the assertion (a). The assertion (b) follows from (1) (a), (3) (b) and (7) (b) we derive

$$\bar{\nabla}_\xi U = (\bar{\nabla}_\xi f)N + f\bar{\nabla}_\xi N = -fA\xi = -FU = \bar{\nabla}_U\xi,$$
$$[U,\xi] = \bar{\nabla}_U\xi - \bar{\nabla}_\xi U = \bar{\nabla}_U\xi - \bar{\nabla}_U\xi = 0,$$

which prove assertion (c). □

By using the decomposition $T\bar{M} = TM \oplus TM^\perp$, we deduce

$$FX = \alpha X - \eta(AX)N \qquad \text{for all} \quad X \in \Gamma(T\bar{M}),$$

where α is a tensor field of type (1, 1) on M, since $g(FX,N) = -g(X,FN) = -g(X,A\xi) = -\eta(AX)$ for all $X \in \Gamma(T\bar{M})$. By using (5), (6), (7), (13) and (15) we obtain following theorem.

Theorem 2. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then the covariant derivative of a tensors t, a, η and α are given by*

$$\begin{aligned}
&(a)\ (\nabla_X t)Y = g(FX,fY)\xi - g(X,Y)\xi - a(Y)AX + B(X,Y)U + \eta(Y)[\alpha tX + X - \eta(AX)U],\\
&(b)\ (\nabla_X a)Y = B(X,tY) + \eta(Y)\eta(AtX),\\
&(c)\ (\nabla_X \eta)Y = g(Y,\nabla_X\xi),\\
&(d)\ (\nabla_X \alpha)Y = R(\xi,X)Y + B(X,Y)A\xi - \eta(AY)AX \text{ for all } X,Y \in \Gamma(TM)
\end{aligned} \tag{18}$$

respectively, where R is the curvature tensor field of M.

From (5), (6), (14) and (18) (a) we get the following.

Proposition 1. *On a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$, we have for all $X \in \Gamma(TM)$*

$$(a)\ \nabla_X U = -tAX + \eta(AtX)\xi, \quad (b)\ B(X,U) = a(AX). \tag{19}$$

Theorem 3. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. The tensor field t is a parallel with respect to the Levi Civita connection ∇ on M iff for all $X \in \Gamma(TM)$*

$$(a)\ AX = \eta(AX)\xi - a(X)\xi + a(AX)U, \quad (b)\ FX = fX - \eta(AX)N + a(X)N. \tag{20}$$

Proof. Suppose that the tensor field t is parallel with respect to ∇, that is $\nabla t = 0$. By using (2) (a), we deduce for all $X, Y \in \Gamma(TM)$

$$\eta(Y)[\alpha tX + X - \eta(AX)U] - a(Y)AX + g(FX,fY)\xi + B(X,Y)U - g(X,Y)\xi = 0. \tag{21}$$

Take $Y = U$ in (21) and using (7) (b), (8), (19) (b) we infer

$$\begin{aligned}
&\eta(U)[\alpha tX + X - \eta(AX)U] - a(U)AX + g(FX,fU)\xi - g(X,U)\xi + B(X,U)U = 0,\\
&\eta(U) = 0, \quad a(U) = 1, \quad g(X,N) = 0,\\
&-AX + g(FX,fU)\xi - g(X,U)\xi + a(AX)U = 0,\\
&AX = g(FX,-N)\xi - a(X)\xi + a(AX)U\\
&\quad = g(X,FN)\xi - a(X)\xi + a(AX)U = g(X,A\xi)\xi - a(X)\xi + a(AX)U,\\
&AX = \eta(AX)\xi - a(X)\xi + a(AX)U
\end{aligned}$$

and the assertion (20) (a) is proved. Next let $Y = fZ$, $Z \in \Gamma(D)$ in (21) and using (1) (f), (3) (b), (17), (20) (a), we deduce for all $X \in \Gamma(TM)$

$$g(X,FZ) = 0 \Rightarrow FX = fX - \eta(AX)N + a(X)N.$$

The proof is complete. □

Proposition 2. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have the assertions for all $X, Y \in \Gamma(TM)$*

$$(a)\ (\nabla_X a)Y = 0 \Leftrightarrow \nabla_X U = 0, \qquad (b)\ (\nabla_X \eta)Y = 0 \Leftrightarrow \nabla_X \xi = 0.$$

Proof. Let $X, Y \in \Gamma(TM)$. Using (8), (16) (b), (18) (b) and (19) (a) we obtain

$$\begin{aligned}
g(\nabla_X U, Y) &= g(-tAX + \eta(AtX)\xi, Y) = g(-tAX, Y) + \eta(AtX)g(\xi, Y)\\
&= g(AX, tY) + \eta(AtX)\eta(Y) = (\nabla_X a)Y,
\end{aligned}$$

which proves assertion (a). The assertion (b) is consequence of the fact that ξ is not a killing vector field. □

According to Theorem 2 in [6], the tensor field $\bar{f} = t + \eta \otimes U - a \otimes \xi$ defines an almost complex structure on M. Moreover, from Proposition 2 we deduce the following assertion.

Theorem 4. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. If the tensor fields t, a, η are parallel with respect to the connection ∇, then $\bar{f}$ defines a Kahler structure on M.*

3 Integrability of distributions on a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$

In this section we establish conditions for the integrability of all distributions on a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$. From Lemma 3 we obtain.

Corollary 1. *On a hypersurface of a quarter symmetric non metric connection M of a quasi-Sasakian manifold $\bar{M}$ there exists a 2-dimensional foliation determined by the integral distribution $D^\perp \oplus (\xi)$.*

Theorem 5. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have the following.*

(a) A leaf of $D^\perp \oplus (\xi)$ is totally geodesic submanifold of M if and only if

$$(1)\ AU = a(AU)U + \eta(AU)\xi - \xi \quad \text{and} \quad (2)\ FN = a(FN)U.$$

(b) A leaf of $D^\perp \oplus (\xi)$ is totally geodesic submanifold of $\bar{M}$ if and only if for all $X \in \Gamma(D)$

$$(1)\ AU = 0 \quad \text{and} \quad (2)\ a(FX) = a(FN) - 1 = 0.$$

Proof. (a) Let M^* be a leaf of integrable distribution $D^\perp \oplus (\xi)$ and h^* be the second fundamental form of the immersion $M^* \to M$. By using (1) (f) and (7) (b) we get for all $X \in \Gamma(TM)$

$$\begin{aligned} g(h^*(U,U),X) &= g(\bar\nabla_U U, X) = -g(N,(\bar\nabla_U f)X - g(\bar\nabla_U N, fX) \\ &= 0 - g(-AU, fX) = g(AU, fX) = g(AU, fX) \end{aligned} \tag{22}$$

and for all $X \in \Gamma(TM)$

$$g(h^*(U,\xi),X) = g(\bar\nabla_U \xi, X) = g(-FU + U, X) = g(FN, fX) + a(X), \tag{23}$$

because $g(FU,N) = 0$ and $f\xi = 0$ the assertion (a) follows from (22) and (23).

(b) Let h_1 be the second fundamental form of the immersion $M^* \to M$. It is easy to see that

$$h_1(X,Y) = h^*(X,Y) + B(X,Y)N \qquad \text{for all} \quad X,Y \in \Gamma(D^\perp \oplus (\xi)). \tag{24}$$

From (6) and (8) we deduce

$$(h_1(U,U),N) = g(\bar\nabla_U U, N) = a(AU), \tag{25}$$

$$g(h_1(U,\xi),N) = g(\bar\nabla_U \xi, N) = a(FN) - 1. \tag{26}$$

The assertion (b) follows from (23)–(26). □

Theorem 6. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar M$. Then*

(a) the distribution $D \oplus (\xi)$ is integrable iff for all $X,Y \in \Gamma(D)$

$$g(AfX + fAX, Y) = 0, \tag{27}$$

(b) the distribution D is integrable iff (27) *holds and for all $X \in \Gamma(D)$*

$$FX = \eta(AtX)U - \eta(AX)N, \quad (\textit{equivalent with } FD \perp D),$$

(c) the distribution $D \oplus D^\perp$ is integrable iff $FX = 0$ for all $X \in \Gamma(D)$.

Proof. Let $X,Y \in \Gamma(D)$. Since ∇ is a torsion free and ξ is a Killing vector field, we infer

$$g([X,\xi],U) = g(\bar\nabla_X \xi, U) - g(\bar\nabla_\xi X, U) = g(\nabla_X \xi, U) + g(\nabla_U \xi, X) = 0. \tag{28}$$

Using (1) (a), (7) (a) we deduce for all $X,Y \in \Gamma(D)$

$$\begin{aligned} g([X,Y],U) &= g(\bar\nabla_X Y - \bar\nabla_Y X, U) = g(\bar\nabla_X Y - \bar\nabla_Y X, fN) \\ &= g(\bar\nabla_Y fX - \bar\nabla_X fY, N) = -g(AfX + fAX, Y). \end{aligned} \tag{29}$$

Next by using (4), (5) (d) and the fact that ∇ is a metric connection we get for all $X,Y \in \Gamma(D)$

$$g([X,Y],\xi) = g(\bar\nabla_X Y, \xi) - g(\bar\nabla_Y X, \xi) = 2g(FX - fX, Y) = 2g(FX,Y) - 2g(fX,Y). \tag{30}$$

The assertion (a) follows from (28), (29) and assertion (b) follows from (28)–(30). Using (6) and (3) we obtain for all $X \in \Gamma(D)$

$$g([X,U],\xi) = g(\bar\nabla_X U, \xi) - g(\bar\nabla_U X, \xi) = 2g(FX,U) - 2g(fX,U). \tag{31}$$

Taking into account that for all $X \in \Gamma(D)$

$$g(FX,N) = g(FfX, fN) = g(FfX, U), \tag{32}$$

the assertion (c) follows from (30) and (31). □

Theorem 7. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar M$. Then we have*

(a) the distribution D is integrable and its leaves are totally geodesic immersed in M if and only if for all $X \in \Gamma(D)$

$$FD \perp D \quad \text{and} \quad AX = a(AX)U - \eta(AX)\xi, \tag{33}$$

(b) the distribution $D \oplus (\xi)$ is integrable and its leaves are totally geodesic immersed in if and only if for $X \in \Gamma(D)$ takes place $AX = a(AX)U$ and $FU = 0$,

(c) the distribution $D \oplus D^\perp$ is integrable and its leaves are totally geodesic immersed in M if and only if for $X \in \Gamma(D)$ takes place $FX = 0$.

Proof. Let M_1^* be a leaf of integrable distribution D and h_1^* the second fundamental form of immersion $M_1^* \to M$. Then by direct calculation we infer

$$g(h_1^*(X,Y),U) = g(\bar\nabla_X Y, U) = -g(Y, \nabla_X U) = -g(AX, tY) \tag{34}$$

and for all $X,Y \in \Gamma(D)$

$$g(h_1^*(X,Y),\xi) = g(\bar\nabla_X Y, \xi) = g(FX,Y) - g(fX,Y). \tag{35}$$

Now suppose M_1^* is a totally submanifold of M. Then (33) follows from (34) and (35). Conversely suppose that (33) is true. Then using the assertion (b) in Theorem 6 it is easy to see that the distribution D is integrable. Next the proof follows by using (34) and (35). Next, suppose that the distribution $D \oplus (\xi)$ is integrable and its leaves are totally geodesic submanifolds of M. Let M_1 be a leaf of $D \oplus (\xi)$ and h_1 the second fundamental form of immersion $M_1 \to M$. By direct calculations, using (6), (7) (b), (16) (b) and (19) (c), we deduce that for all $X,Y \in \Gamma(D)$

$$g(h_1(X,Y),U) = g(\bar\nabla_X Y, U) = -g(AX, tY), \tag{36}$$

and for all $X \in \Gamma(D)$

$$g(h_1(X,\xi),U) = g(\bar\nabla_X \xi, U) = g(-FU + fU, X) = g(FU, X). \tag{37}$$

Then the assertion (b) follows from (32), (36), (37) and the assertion (a) of Theorem 6. Next let $\bar M_1$ be a leaf of the integrable distribution $D \oplus D^\perp$ and $\bar h_1$ is the second fundamental form of the immersion $M_1 \to M$. By direct calculation for all $X \in \Gamma(D), Y \in \Gamma(D \oplus D^\perp)$ we get

$$g(\bar h_1(X,Y),\xi) = g(FX,Y) - g(fX,Y). \tag{38}$$

The assertion (c) follows from (3) (c), (32) and (38). □

4 Contact conformal connection on a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar M$

Let the conformal change of the metric tensor $\bar g$ which induces a new metric tensor, given by $\bar{\bar g}(X,Y) = e^{2p}\bar g(X,Y)$ with regard to this metric, take an affine connection, which satisfies

$$\bar{\bar\nabla}_X \bar{\bar g}(Y,Z) = \bar\nabla_X\{e^{2p}\bar g(Y,Z)\} = e^{2p}p(X)\eta(Y)\eta(Z), \tag{39}$$

where p is a scalar point function. The torsion tensor of the connection $\bar{\bar{\nabla}}$ satisfies

$$T(X,Y) = -2\bar{g}(fX,Y)U = S(X,Y) - S(Y,X), \tag{40}$$

where U is a vector field. Let

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + S(X,Y), \tag{41}$$

where S is a tensor of type $(1,2)$. Using (39), (40), (41), we have

$$\begin{aligned}\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + p(X)\{Y - \eta(Y)\xi\} + p(Y)\{X - \eta(X)\xi\}\\ - \bar{g}(fX, fY)P + u(X)fY + u(Y)fX - \bar{g}(fX,Y)U,\end{aligned} \tag{42}$$

where $\bar{g}(P,X) = p(X)$, $\bar{g}(QX,P) = p(fX) = -q(X)$, $\bar{g}(Q,X) = q(X)$, $\bar{g}(U,X) = u(X)$.

$$\begin{aligned}(\bar{\bar{\nabla}}_X f)(Y) = (\bar{\nabla}_X f)(Y) + \{X - \eta(X)\xi\}p(fY) - p(Y)fX + \bar{g}(fX,Y)p + \bar{g}(fX,fY)fP\\ + u(fY)fX + u(Y)\{X - \eta(X)\xi\} - \bar{g}(fX,fY)U + \bar{g}(fX,Y)fU = 0.\end{aligned}$$

Using (5), the relation becomes

$$\begin{aligned}\bar{g}(\bar{\bar{\nabla}}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi + \eta(Y)X - \eta(X)\eta(Y)\xi - p(Y)fX\\ + \{X - \eta(X)\xi\}p(fY) + \bar{g}(fX,Y)p + \bar{g}(fX,fY)fP + u(fY)fX\\ + u(Y)\{X - \eta(X)\xi\} - \bar{g}(fX,fY)U + \bar{g}(fX,Y)fU = 0.\end{aligned}$$

Contracting with respect to X,

$$\begin{aligned}2m\eta(Y) + 2mp(fY) - 2p(fY) + 2mu(Y) - 2u(Y) + 2\eta(U)\eta(Y) = 0,\\ 2(m-1)p(fY) + 2(m-1)u(Y) + 2\eta(Y)\{m + \eta(U)\} = 0.\end{aligned}$$

If we put $\eta(U) = -1 = u(\xi)$, then $u(Y) = q(Y) - \eta(Y)$. Thus (42) takes the form

$$\begin{aligned}\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \{Y - \eta(Y)\xi\}p(X) + \{X - \eta(X)\xi\}p(Y) - \bar{g}(fX,fY)P\\ + \{q(X) - \eta(X)\}fY + \{q(Y) - \eta(Y)\}fX - \bar{g}(fX,Y)(Q - \xi).\end{aligned} \tag{43}$$

Then $\bar{\bar{\nabla}}_X\xi = 0 = \bar{\nabla}_X\xi + \{X - \eta(X)\xi\}p(\xi) - fX$. Using (6) in this equation, we have

$$-FX + fX + \bar{\nabla}_X\xi + \{X - \eta(X)\xi\}p(\xi) - fX = 0,$$

which implies that $FX = \{X - \eta(X)\xi\}p(\xi)$.

Proposition 3. *On a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$ the affine connection $\bar{\bar{\nabla}}$ which satisfies* (40)*, is given by* (43) *with the conditions* $u(\xi) = -1 = \eta(U)$, $FX = \{X - \eta(X)\xi\}p(\xi)$.

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Рахман Ш. *Геометрія гіперповерхонь четвертинно симетричного неметричного зв'язку в квазі Сасакяновому многовиді* // Карпатські матем. публ. — 2015. — Т.7, №2. — C. 226–235.

Метою цієї статті є вивчення поняття CR-підмноговидів та існування деяких структур на гіперповерхні четвертинно симетричного неметричного зв'язку в квазі Сасакяновому многовиді. Ми досліджуємо існування структури Кахлера на M та існування глобально метричної конструкції f-структури у сенсі Гольдберга С.І., Яно К. [6]. Обговорюється інтегрованість розподілів на M і геометрія їхніх листків. Описано спроби пов'язати цей результат з отриманими раніше результатами Гольдберга В., Роска Р., які присвячені многовиду Сасакяна та конформним зв'язкам.

Ключові слова і фрази: CR-підмноговид, квазі Сасакяновий многовид, четвертинно симетричний неметричний зв'язок, умови інтегрованості розподілів.

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A STUDY ON INTEGER ADDITIVE SET-VALUATIONS OF SIGNED GRAPHS

Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-labeling (IASL) of a graph G is an injective set-valued function $f : V(G) \to \mathcal{P}(\mathbb{N}_0) \setminus \{\varnothing\}$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0) \setminus \{\varnothing\}$ is defined by $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sumset of $f(u)$ and $f(v)$. A graph which has an IASL is usually called an IASL-graph. An IASL f of a graph G is said to be an integer additive set-indexer (IASI) of G if the associated function f^+ is also injective. In this paper, we define the notion of integer additive set-labeling of signed graphs and discuss certain properties of signed graphs which admits certain types of integer additive set-labelings.

Key words and phrases: signed graphs, balanced signed graphs, clustering of signed graphs, IASL-signed graphs, strong IASL-signed graphs, weak IASL-signed graphs, isoarithmetic IASL-signed graphs.

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1 INTRODUCTION

For all terms and definitions, not defined specifically in this paper, we refer to [3, 5, 16] and for the topics in signed graphs we refer to [17, 18]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

1.1 An overview of IASL-graphs

The *sum set* (see [7]) of two sets A and B, denoted by $A + B$, is defined as $A + B = \{a + b : a \in A, b \in B\}$. Let $\mathbb{N}_0$ be the set of all non-negative integers and let X be a non-empty subset of $\mathbb{N}_0$. Using the concepts of sumsets, we have the following notions as defined in [4, 8].

An *integer additive set-labeling* (shortly IASL) is an injective function $f : V(G) \to \mathcal{P}(X) \setminus \{\varnothing\}$ such that the induced function $f^+ : E(G) \to \mathcal{P}(X) \setminus \{\varnothing\}$ is defined by

$$f^+(uv) = f(u) + f(v)$$

for all $u, v \in E(G)$. A graph G which is endowed with an IASL is called an *integer additive set-labeled graph* (IASL-graph).

An *integer additive set-indexer* (IASI) is an injective function $f : V(G) \to \mathcal{P}(X) \setminus \{\varnothing\}$ such that the induced function $f^+ : E(G) \to \mathcal{P}(X) \setminus \{\varnothing\}$ is also injective. A graph G which is endowed with an IASI is called an *integer additive set-indexed graph* (IASI-graph).

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An IASL (or IASI) is said to be *k-uniform* if $|f^+(e)| = k$ for all $e \in E(G)$. That is, a connected graph G is said to have a k-uniform IASL (or IASI) if all of its edges have the same set-indexing number k. The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element. If the set-labels of all vertices of G have the same cardinality, then the vertex set $V(G)$ is said to be *uniformly set-indexed*. An element is said to be *mono-indexed* if its set-indexing number is 1.

A *weak integer additive set-indexer* of a graph G is an IASI $f : V(G) \to \mathcal{P}(X) \setminus \{\varnothing\}$ such that

$$|f^+(uv)| = \max(|f(u)|, |f(v)|)$$

for all $u, v \in V(G)$ and a *strong integer additive set- indexer* (SIASI) of G is an IASI such that

$$|f^+(uv)| = |f(u)|\,|f(v)|$$

for all $u, v \in V(G)$.

The following result is a necessary and sufficient condition for a graph to be a weak IASL-graph.

Lemma 1 ([10]). *An IASI f of a given graph G is a weak IASI of G if and only if at least one end vertex of every edge of G is mono-indexed, with respect to f.*

Theorem 1 ([8]). *A graph G admits a weakly uniform IASL if and only if G is bipartite.*

An integer additive set-indexer $f : V(G) \to \mathcal{P}(X) \setminus \{\varnothing\}$ of a graph G, satisfying the condition $|f^+(uv)| = |f(u) + f(v)| = |f(u)|\,|f(v)|$ for all edges uv of G, is said to be a *strong IASI* of G. A graph which has a strong IASI is called a *strong IASI-graph*.

Theorem 2 ([9]). *A graph G admits a strong IASI, say f, if and only if for any two adjacent vertices in G, the sets defined by $D_{f(u)} = \{|a - b| : a, b \in f(u)\}$ and $D_{f(v)} = \{|c - d| : c, d \in f(v)\}$ are disjoint.*

Theorem 3 ([9]). *A connected graph G admits a strongly k-uniform IASL if and only if either G is bipartite or k is a perfect square.*

An IASL f of a given graph G is called an arithmetic IASL of G if the elements of the set-labels of vertices and edges of G are in arithmetic progressions. If all these arithmetic progressions have the same common difference d, then such an arithmetic IASL is called *isoarithmetic IASL* of G.

Theorem 4 ([11]). *If $f : V(G) \to \mathcal{P}(X)$ is an isoarithmetic IASL defined on a graph G, then the cardinality of the set-label of any edge uv in G is $|f(u)| + |f(v)| - 1$.*

1.2 Preliminaries on Signed Graphs

Note that a half edge of a graph G is an edge having only one end vertex and a loose edge of G is an edge having no end vertices.

A *signed graph* (see [17, 18]), denoted by $\Sigma(G, \sigma)$, is a graph $G(V, E)$ together with a function $\sigma : E(G) \to \{+, -\}$ that assigns a sign, either $+$ or $-$, to each ordinary edge in G. The function σ is called the *signature* or *sign function* of Σ, which is defined on all edges except half edges and is required to be positive on free loops.

An edge e of a signed graph Σ is said to be a *positive edge* if $\sigma(e) = +$ and an edge $\sigma(e)$ of a signed graph Σ is said to be a *negative edge* if $\sigma(e) = -$. The set E^+ denotes the set of all positive edges in Σ and the set E^- denotes the set of negative edges in Σ. A simple cycle (or path) of a signed graph Σ is said to be *balanced* (see [2, 6]) if the product of signs of its edges is $+$. A signed graph is said to be a *balanced signed graph* if it contains no half edges and all of its simple cycles are balanced.

Note that a negative signed graph is balanced if and only if it is bipartite.

Balance or imbalance is the main property of a signed graph. The following theorem, well known as *Harary's Balance Theorem*, establishes a criteria for balance in a signed graph.

Theorem 5 ([6]). *The following statements about a signed graph are equivalent.*

i) A signed graph Σ is balanced.

ii) Σ has no half edges and there is a partition (V_1, V_2) of $V(\Sigma)$ such that $E^- = E(V_1, V_2)$.

iii) Σ has no half edges and any two paths with the same end points have the same sign.

A signed graph Σ is said to be *clusterable* or *partitionable* (see [17, 18]) if its vertex set can be partitioned into subsets, called *clusters*, so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters. If $V(\Sigma)$ can be partitioned in to k subsets with the above mentioned conditions, then the signed graph Σ is said to be *k-clusterable*. In this paper, we discuss only the 2-clusterability of signed graphs.

It can be noted that 2-clusterability always implies balance in a signed graph Σ. The converse need not be true. If all edges in Σ are positive edges, then Σ is balanced but not 2-clusterable.

In this paper, we extend the studies on different types of integer additive set-labeling of graphs to classes of signed graphs and hence study the properties and characteristics of such signed graphs.

2 IASL-SIGNED GRAPHS

Motivated from the studies on set-valuations of signed graphs in [1], and the studies on integer additive set-labeled graphs in [4, 8, 9, 11], we define the notion of an integer additive set-labeling of signed graph as follows.

Definition 1. *Let $X \subseteq \mathbb{N}_0$ and let Σ be a signed graph, with corresponding underlying graph G and the signature σ. An injective function $f : V(\Sigma) \to \mathcal{P}(X) \setminus \{\varnothing\}$ is said to be an integer additive set-labeling (IASL) of Σ if f is an integer additive set-labeling of the underlying graph G and the signature of Σ is defined by $\sigma(uv)) = (-1)^{|f(u)+f(v)|}$.*

A signed graph which is endowed with an integer additive set-labeling is called an integer additive set-labeled signed graph (IASL-signed graph) and is denoted by Σ_f.

Definition 2. *An integer additive set-labeling f of a signed graph Σ is said to be an integer additive set-indexer of Σ if f is an integer additive set-indexer of the underlying graph G.*

Definition 3. *An IASL f of a signed graph Σ is called a weak IASL or a strong IASL or an arithmetic IASL of Σ, in accordance with the IASL f of the underlying graph G is a weak IASL or a strong IASL or an arithmetic IASL of the corresponding underlying graph G.*

The structural properties and characteristics of different types of IASL-signed graphs are interesting. In the following section, we study the properties of strong IASL-signed graphs.

2.1 Strong IASL-signed graphs

As stated earlier, balance is the fundamental characteristic of a signed graph and hence let us investigate the conditions required for a strong IASL-signed graph to have the balance property. The following result provides a necessary and sufficient condition for the existence of a balanced signed graph corresponding to a strongly uniform IASL-graph.

Theorem 6. *A strongly k-uniform IASL-signed graph Σ is balanced if and only if the underlying graph G is a bipartite graph or $\sqrt{k}$ is an even integer.*

Proof. Assume that the strongly k-uniform IASL-signed graph Σ is balanced. Then, for any cycle C_r, $\sigma(C_r)$ must be positive. Let n_1 and n_2 be two positive integers greater than 1 such that $n_1n_2 = k$. Label the vertices of C_r alternatively by n_1-element subsets and n_2-element subsets of the ground set X. Here we have the following cases.

Case 1. Let $n_1 \neq n_2$. We claim that this labeling is possible only when C_r is even. If C_r is an odd cycle, then by labeling the vertices of C_r as mentioned above, there will be two adjacent vertices, say u and v both having n_1-element set-labels or n_2-element set-labels and the edge uv has the set-indexing number n_1^2 (or n_2^2)), which is a contradiction to the fact that G is strongly k-uniform IASL-graph. Therefore, G is bipartite.

Case 2. Let C_r be an odd cycle in G. Then, we claim that the above mentioned labeling is possible only when $n_1 = n_2$. If C_r is an odd cycle, as mentioned in Case-1, there exists an edge in C_r with set-indexing number n_1^2 (or n_2^2). Since G admits a strongly k-uniform IASL, we have $n_1^2 = k = n_1n_2$. This is true only if $n_1 = n_2 = \sqrt{k}$. That is, k is a perfect square. Therefore, every vertex of C_r has the set-indexing number $\sqrt{k}$ and every edge of C_r has the set-indexing number k. Since Σ is balanced, we have $\sigma(C_r) = +$, which is possible when k and hence $\sqrt{k}$ are even integers.

Conversely, assume that the underlying graph G of strongly k-uniform IASL-signed graph Σ is a bipartite graph or $\sqrt{k}$ is an even integer. Then, consider the following cases.

Case 1. Assume that the underlying graph G is a bipartite graph. Then, G has no odd cycles. Let C_r be an arbitrary cycle in G, where r is an even integer. Consider $n_1, n_2 \in \mathbb{N}_0$ such that $n_1n_2 = k$. Now, label the vertices of G alternatively by n_1-element subsets and n_2 element subsets of the ground set X such that the labeling becomes a strongly k-uniform IASL of G. Here, we have the following subcases.

Subcase 1.1. If both n_1 and n_2 are odd integers, then k is odd and hence every edges of the cycle C_r in Σ has the negative sign. Since, C_r has even number of edges, we have $\sigma(C_r) = +$.

Subcase 1.2. If one or both of n_1 and n_2 is even, then k is even and every edge of C_r has the positive sign. Therefore, $\sigma(C_r) = +$.

Case 2. Assume that G is a non-bipartite graph. By hypothesis, $\sqrt{k}$ is an even integer and hence k is also an even integer. Since G is not bipartite, it contains odd cycles. Let C_r be an arbitrary odd cycle in G. Since G admits a strongly k-uniform IASL, every edge of C_r must be labeled by the subsets of X having cardinality $\sqrt{k}$ (see [9]). Therefore, every edge of G has positive sign and hence $\sigma(C_r) = +$.

In all above cases, we can see that strongly k-uniform IASL-signed graph Σ is balanced. □

What are the conditions required for a strongly uniform IASL-signed graph to be clusterable? The following result provides a solution to this problem.

Proposition 1. *A strongly k-uniform IASL-signed graph Σ is clusterable if and only if the underlying graph G is bipartite and k is an odd integer.*

Proof. Let Σ is clusterable. Let (U_1, U_2) be a partition of $V(\Sigma)$ with the required properties of a clustering of Σ. Clearly, k must be odd. For, if k is even, all edges of Σ will be positive edges and hence all vertices of Σ belong to either U_1 or to U_2 making other empty, contradicting the fact that Σ is clusterable. As k is odd, every edge of Σ is a negative edge and hence for any two adjacent vertices in Σ must belong to different partitions. Choose vertices which are pairwise non-adjacent in Σ to form a subset U_1 of $V(\Sigma)$ and let $U_2 = V \setminus U_1$. Clearly, U_2 is also a subset of V in which vertices are pairwise disjoint. Therefore, (U_1, U_2) is a bipartition of the underlying graph G. Hence G is bipartite.

Conversely, assume that the underlying graph G of a strongly k-uniform IASL-signed graph Σ is a bipartite graph with bipartition (V_1, V_2) and k is an odd integer. Therefore, every edge of Σ is a negative edge with one end in V_1 and other end in V_2. Then, (V_1, V_2) satisfies the properties of a clustering for Σ. Hence, Σ is clusterable. □

If the underlying graph G of a strong IASL-signed graph Σ is bipartite, then Σ is balanced if and only if the number of negative edges in every cycle of G in Σ is even. This is possible only when the number of distinct pairs of adjacent vertices, having odd parity set-labels, in every cycle of Σ is even. Therefore, we have the following assertion.

Proposition 2. *Let Σ be a strong IASL-signed graph with the underlying graph G bipartite. Then, Σ is clusterable if and only if the number of distinct pairs of adjacent vertices having odd parity set-labels is even.*

The proof of the above theorem is very obvious. The following result describes the conditions required for the clusterability of (non-uniform) strong IASL-signed graphs whose underlying graph G is a bipartite graph.

Proposition 3. *The strong IASL-signed graph, whose underlying graph G is a bipartite graph, is clusterable if and only if there exist at least two adjacent vertices in Σ_f with odd parity set-labels.*

Proof. Let Σ be a strong IASL-signed graph whose underlying graph G is a bipartite graph. Then, the same IASL of Σ is a strong IASL of G also. Since G is bipartite, every cycle in G is an even cycle. Let $C_r : v_1v_2v_3 \ldots v_rv_1$ be a cycle in G.

First assume that Σ is clusterable. Then, there exists a partition (U_1, U_2) of non-empty subsets of $V(\Sigma)$ such that the edges connecting vertices in the same partition have positive sign and the edges connecting vertices in the different partitions have the negative sign. Note that an edge uv of G has a negative sign only when both u and v have odd parity set-labels. Since G is a connected graph and both sets U_1 and U_2 are non-empty, there must be at least one edge, say $e = uv$, in G with one end vertex in U_1 and the other end vertex in U_2 such that both u and v have odd parity set-labels.

Conversely, assume that at least two adjacent vertices of G have odd parity set-labels. If u and v be two vertices of G having odd cycles in G. Then, $\sigma(uv) = -$ in Σ. Let U_1 and U_2 be two mutually exclusive subsets of $V(G)$ such that $u \in U_1$ and $v \in U_2$. If there exist other edges, say xy such that both x and y have odd parity set-labels, then include any one of x and y to U_1 and all other vertices to U_2. Repeat this process until all adjacent pairs of vertices having odd parity set-labels are counted. Then, (U_1, U_2) will be a partition of Σ with desired properties. Therefore, Σ is clusterable. □

Let Σ be a strong IASL-signed graph, whose underlying graph G is a non-bipartite graph. Then G contains some odd cycles. If C_r is an arbitrary odd cycle in G, then Σ is balanced if and only if the number of negative edges in C_r is even, which is possible only when the number of positive edges in C_r is odd. It is possible only when at least two adjacent vertices must have even parity set-labels. A necessary and sufficient condition for a strong IASL-signed graph to be clusterable is described in the following theorem.

Theorem 7. *A strong IASL-signed graph Σ is clusterable if and only if every odd cycle of the underlying graph G has at least two adjacent vertices with even parity set-label and at least two adjacent vertices with odd parity set-label.*

Proof. Let Σ be a strong IASL-signed graph with underlying graph G, where G is a non-bipartite graph. Then, G contains odd cycles. Let $C_r : v_1v_2 \ldots v_rv_1$ be a cycle of length r in G. If Σ is clusterable, there exists a partition of vertices (U_1, U_2) such that all edges having end vertices in the same partition have positive sign and the edges having end vertices in the different partitions have negative sign. If all vertices of Σ have even parity set-labels, then all edges of Σ will be positive edges. Hence, all vertices of Σ must belong to the same partition, say U_1, making the other partition, say U_2 empty. If one end vertex of every edge of Σ has even parity set-label, then also all edges of Σ become positive edges. In this case also, all vertices of Σ are in the same partition and the other partition is empty. Hence, there must be at least one edge in Σ such that its both end vertices have odd parity set-labels.

Conversely, assume that every odd cycle, say C_r, contains two adjacent vertices having even parity set-labels. Without loss of generality, let v_1 and v_2 be the vertices in C_r, which have even parity set-labels. Let $v_1, v_2 \in U_1$. Let all other vertices have odd parity set-labels. Since $\sigma(v_2v_3) = +$, v_3 must also be an element of U_1. Since $\sigma(v_3v_3) = -$, $v_3 \in U_2$. Proceeding like this, we have $v_4, v_6, \ldots, v_{r-1}$ are in U_2 and $v_5, v_7, \ldots, v_r$ are in U_1. This partition (U_1, U_2) is a clustering for Σ. □

In this context, the questions on the balance and clusterability of weak IASL-signed graphs arouse much interest. In the following section, we discuss certain properties of weak IASL-signed graphs that are similar to those of strong IASL-signed graphs.

2.2 Weak IASL-signed graphs

Balance and clusterability of the induced signed graphs of weak IASL-graphs has been described in the following theorems. Analogous to Proposition 6, the balance of weakly uniform IASL-signed graph can be described as follows.

Proposition 4. *A weakly k-uniform IASL-signed graph is always balanced.*

Proof. Let Σ be a weakly k-uniform IASL-signed graph with the underlying graph G, where k is any positive integer greater than 1. Then, G admits a weakly k-uniform IASL, say f, then $f^+(e) = k$ for all $e \in G$. Then, the signature σ is given by $\sigma(e) = (-1)^k$ for all $e \in \Sigma$. Therefore, the signs of all edges of Σ are all odd or all even. Since the underlying graph G is a weakly k-uniform IASL-graph, then G is bipartite (see [4]). Therefore, G has no odd cycles. Therefore, the number of signs, positive or negative, of edges in each cycles are even. Therefore, for any cycle C_r in Σ, $\sigma(C_r)$ is positive. Hence, Σ is balanced. □

The following theorem discusses the clusterability of weakly uniform IASL-signed graphs.

Theorem 8. *A weakly k-uniform IASL-signed graph Σ is clusterable if and only if k is a positive odd integer.*

Proof. Let the given weakly k uniform IASL-signed graph Σ be clusterable. Then, there exists a partition (U_1, U_2) of non-empty subsets of $V(\Sigma)$ such that the end vertices of positive edges belongs to the same partition and the end vertices of negative edges belong to different partitions. If k is even, then all edges in Σ are positive edges and all vertices in Σ belong to the same partition, say U_1. Therefore, $U_2 = \emptyset$, which contradicts the hypothesis that Σ is clusterable. Hence, k is an odd integer.

Conversely, assume that k is an odd integer. Then, every edge of G is a negative edge. Then, the bipartition (V_1, V_2) of the underlying graph G, where V_1 is the set of all mono-indexed vertices and V_2 is the set of all vertices having set-indexing number k, will form a 2-clustering of Σ. That is, Σ is clusterable. □

Balance of a weak IASL-signed graph whose underlying graph is a bipartite graph is discussed in the following result.

Proposition 5. *A weak IASL-signed graph Σ, whose underlying graph G is a bipartite graph, is balaced if and only if the number of odd parity non-singleton set-labels in every cycle of Σ is even.*

Proof. Assume that a weak IASL-signed graph Σ is balanced. Note that for the corresponding underlying graph G, the set-indexing number of every edge, not mono-indexed, is the cardinality of the non-singleton set-label of its end vertex. Hence, for every odd parity non-singleton vertex set-labels in Σ, the corresponding edge has a negative sign. Hence, for any cycle C_r in Σ we have $\sigma(C_r) = +$ and this is possible only when the number of odd parity non-singleton vertex set-labels in C_r of Σ is even.

Conversely, assume that the number of odd parity non-singleton vertex set-labels in any cycle of Σ is even. Therefore, the number of negative edges in Σ is even. Hence, for any cycle C_r in Σ, $\sigma(C_r) = +$ and hence Σ is balanced. □

The following theorem establishes a necessary and sufficient condition for a weak IASL-signed graph whose underlying graph is a bipartite graph.

Theorem 9. *The weak IASL-signed graph Σ, whose underlying graph G is a bipartite graph, is clusterable if and only if there exist some non-singleton vertex set-labels of Σ which are odd parity sets.*

Proof. Let G be a bipartite graph with bipartition (V_1, V_2). Then G need not have any mono-indexed edge. Then, without loss of generality, let V_1 contains all mono-indexed vertices and V_2 contains all vertices having non-singleton set-labels. Let Σ denotes the corresponding induced signed graph Σ of G.

Assume that some set-labels of the vertices in V_2 are odd parity sets. Since the mono-indexed vertices in G are not adjacent in G, all vertices in V_1 can be in the same cluster U_1, if exists. Then, by the definition of clustering, the vertices having even parity set-labels cannot be included in the second cluster U_2 as the signs of edges connecting these vertices to the vertices in V_1 are positive. Therefore, let $U_1 = V_1 \cup V_2'$ and $U_2 = V_2 - V_2'$, where V_2' is the proper subset of all vertices in V_2 having even parity set-labels. Clearly, all the edges in the same partition, if exist, have positive signs and the edges connecting the vertices in different partitions have negative sign. That is, G is clusterable.

Conversely, assume that Σ is clusterable. Then, there exists a partition (U_1, U_2) of the vertex set $V(\Sigma)$ such that all edges connecting the vertices in the same partition have the positive sign and the edges connecting the vertices in different partitions have negative sign. Let U_1 contains all vertices in V_1. Any vertex u in V_2, having an even parity set label and adjacent to some vertex v in V_1 must also belong to U_1 as $\sigma(uv) = +$. Hence, if the set-labels of all vertices in V_2 are even parity sets, then $U_2 = \emptyset$, which is a contradiction to the hypothesis that Σ is clusterable. Therefore, the set-labels of some vertices in V_2 are odd parity sets. □

In this context, it is much interesting to check the balance property of weak IASL-signed graphs whose underlying graphs are non-bipartite. Hence, we have the following theorem.

Theorem 10. *A weak IASL-signed graph Σ, whose underlying graph G is a non-bipartite graph, is not balanced.*

Proof. Since G is a non-bipartite graph, G contains odd cycles. Let C_r be an odd cycle in G. If Σ is balanced, then the number of negative edges in C_r is even. When one vertex, say v, of G has an even parity set-label, then the two edges incident on it will have the positive sign and the remaining odd number of edges in C_r have negative signs. If u and v are two adjacent vertices in C_r, then the three edges incident on these two vertices become positive and the number of negative edges in C_r becomes even.

Therefore, if G is balanced, then at least two adjacent vertices must have even parity set-labels. This contradicts the fact that G admits a weak IASL. Hence, Σ is not balanced. □

The following result is a straight forward implication of the above theorem.

Corollary 1. *A weak IASL-signed graph Σ, whose underlying graph G is non-bipartite, is not clusterable.*

Proof. For any signed graph Σ, we have 2-clusterability implies the balance in Σ. But by Theorem 10, a weak IASL-signed graph Σ, whose underlying graph is non-bipartite, can not be a balanced signed graph. Hence, the weak IASL-signed graph Σ is not clusterable. □

Another interesting type IASL-signed graph is the signed graph which admits an isoarithmetic IASL. In the following section, we discuss the properties of these types of signed graphs.

2.3 Isoarithmetic IASL-signed graphs

The following theorem describes a necessary and sufficient condition for an isoarithmetic IASL-signed graph to be balanced.

Theorem 11. *An isoarithmetic IASL-signed graph Σ is balanced if and only if every cycle in Σ has even number of distinct pairs of adjacent vertices having the same parity set-labels.*

Proof. Let Σ be a an isoarithmetic IASL-signed graph. Then, for every edge uv in $E(\Sigma)$, the cardinality of the set-label of uv is $|f^+(uv)| = |f(u)| + |f(v)| - 1$. Then, $|f^+(uv)|$ is odd if both $f(u), f(v)$ are of the same parity and $|f^+(uv)|$ is even if $f(u), f(v)$ are of different parities.

Assume that Σ is balanced. Then, the number of negative edges in every cycle of Σ is even and the number of disjoint pairs of adjacent vertices having the same parity set-labels is even. Conversely, assume that the number of disjoint pairs of adjacent vertices in every cycle C_r of Σ having the same parity set-labels is even. Then, the number of negative edges in C_r is even. Therefore, Σ is balanced. □

What are the conditions required for an isoarithmetic IASL-signed graph to be clusterable? The following result provides the required conditions in this regard.

Proposition 6. *An isoarithmetic IASL-signed graph Σ is clusterable if and only if Σ contains some disjoint pairs of adjacent vertices having the same parity set-labels.*

Proof. Note that a connected signed graph Σ is clusterable, if and only if Σ must have negative edges connecting the vertices in different partitions. Hence, if an isoarithmetic IASL-graph Σ is clusterable, then Σ contains negative edges which is possible when some disjoint pairs of adjacent vertices in Σ must have the same parity set-labels. □

3 Conclusion

In this paper, we discussed the characteristics and properties of the induced signed graphs of certain IASL-graphs with a prime focus on clusterability and balance of these signed graphs. There are several open problems in this area. Some of the open problems that seem to be promising for further investigations are following.

Problem 1. *Discuss the k-clusterability of different types of IASL-signed graphs for $k > 2$.*

Problem 2. *Discuss the balance, 2-clusterability and general k-clusterability of other types of arithmetic IASL-signed graphs of different types of arithmetic and semi-arithmetic IASL-graphs.*

For $X \subseteq \mathbb{N}_0$, an IASL f of a graph G, is said to be a *topological IASL* of G if $\mathcal{T} = f(V(G)) \cup \{\varnothing\}$ is a topology on X (see [12]) and is said to be a topogenic IASL of G if $\mathcal{T} = f(V(G)) \cup f^+(E(G)) \cup \{\varnothing\}$ is a topology on X (see [13]). Then, we have the following problem.

Problem 3. *Discuss balance and k-clusterability of topological and topogenic IASL-signed graphs.*

An integer additive set-indexer f of a graph G, with respect to a finite set $X \subseteq \mathbb{N}_0$, is said to be a *graceful integer additive set-labeling* of G if $f^+(E(G)) \cup \{\{0\}, \varnothing\} = \mathcal{P}(X)$ (see [14]) and is said to be a *sequential integer additive set-labeling* of G if $f(V(G)) \cup f^+(E(G)) \cup \{\varnothing\} = \mathcal{P}(X)$ (see [15]). Then, we obtain the following problems.

Problem 4. *Discuss the balance and k-clusterability of graceful and sequential IASL-signed graphs.*

Problem 5. *Discuss the balance and general k-clusterability of topologically graceful and topologically sequential IASL-signed graphs.*

Problem 6. *Discuss the balance and k-clusterability of different types of integer additive set-indexed graphs.*

Study on several other IASLs under which the collection of set-labels of given IASL-signed graphs form certain other subset structures like filters of the ground set. The study on certain IASL-signed graphs, where the edge set-labels are the union or intersection of the set-labels of end vertices.

Further studies on other characteristics of signed graphs corresponding to different IASL-graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

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Нехай $\mathcal{P}(\mathbb{N}_0)$ позначає множину підмножин всіх невід'ємних цілих чисел $\mathbb{N}_0$. Цілочисельним адитивним позначенням (IASL) графа G називається така ін'єктивна множинно-значна функція $f : V(G) \to \mathcal{P}(\mathbb{N}_0) \setminus \{\emptyset\}$, що індукована функція $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0) \setminus \{\emptyset\}$ визначена $f^+(uv) = f(u) + f(v)$, де $f(u) + f(v)$ об'єднання множин $f(u)$ і $f(v)$. Граф, який має цілочисельне адитивне позначення (IASL), зазвичай називають IASL-графом. IASL f графа G називають цілочисельно адитивно індексуючим (IASI), якщо асоційована функція f^+ також ін'єктивна. У цій статті ми визначаємо поняття цілочисельно адитивного позначення графів зі знаками та описуємо відповідні властивості цих графів, які мають деякі типи цілочисельного адитивного позначення.

Ключові слова і фрази: графи зі знаками, збалансовані графи зі знаками, кластеризація графів зі знаками, IASL-графи зі знаками, сильні IASL-графи зі знаками, слабкі IASL-графи зі знаками, ізоарифметичні IASL-графи зі знаками.

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ON PROPERTIES OF THE SOLUTIONS OF THE WEBER EQUATION

Growth, convexity and the l-index boundedness of the functions $\alpha(z)$ and $\beta(z)$, such that $\alpha(z^4)$ and $z\beta(z^4)$ are linear independent solutions of the Weber equation $w'' - (\frac{z^2}{4} - \nu - \frac{1}{2})w = 0$ with $\nu = -\frac{1}{2}$ are investigated.

Key words and phrases: entire function, l-index boundedness, convex function, growth, Weber equation.

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Introduction

Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

be an entire function, l — a positive continuous on $[0, +\infty)$ function. Function f is said to be of bounded l-index [3], if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{ \frac{|f^{(k)}(z)|}{k! l^k(|z|)} : 0 \le k \le N \right\}. \tag{2}$$

The least such integer N is called l-index and is denoted by $N(f,l)$. If $G \subset \mathbb{C}$ and there exists $N \in \mathbb{Z}_+$ such that inequality (2) holds for all $n \in \mathbb{Z}_+$ and $z \in G$, analytic in G function f is said to be of bounded l-index on (or in) G, and l-index is denoted by $N(f,l;G)$. Theorem 2.2 [3, p.33] implies that *if f is an entire function, G is a bounded domain and l — a positive continuous function, then f is of bounded l-index in G.*

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function (1) is said to be convex if $f(\mathbb{D})$ is a convex domain. Condition $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0\ (z \in \mathbb{D})$ is necessary and sufficient [1] for a convexity of f. Every convex function is univalent in $\mathbb{D}$, and therefore $f_1 \ne 0$.

Differential equation

$$w'' - \left(\frac{z^2}{4} - \nu - \frac{1}{2}\right) w = 0 \tag{3}$$

is said to be the Weber equation. Properties of the solutions of the Weber equation if $\nu \ne -\frac{1}{2}$ are investigated in [5] and the following theorem is proved.

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Theorem ([5]). *If $\nu \neq -\frac{1}{2}$ the general solution of the equation (3) is of the form $w(z) = C_1\varphi(z^2) + C_2 z\psi(z^2)$, and the functions $\varphi(z)$ and $\psi(z)$ have the following properties:*

1) $N(\varphi, l) \leq 1$ *with* $l(|z|) \equiv \frac{28}{9}|2\nu+1| + \frac{18}{5}$ *and* $N(\psi, l) \leq 1$ *with* $l(|z|) \equiv \frac{11}{10}(|2\nu+1|+2)$;
2) *if* $(762 - \sqrt{388564})/343 \leq |2\nu+1| \leq (762+\sqrt{388564})/343$, *then* $\varphi(z)$ *is convex in* $\mathbb{D}$, *and if* $(2350 - \sqrt{3590164})/639 \leq |2\nu+1| \leq (2350+\sqrt{3590164})/639$ *then* $\psi(z)$ *is convex in* $\mathbb{D}$;
3) *if* $(1623 - \sqrt{2430289})/364 \leq |2\nu+1| \leq (1623+\sqrt{2430289})/364$, *then* $\varphi(z)$ *is close-to-convex in* $\mathbb{D}$, *and if* $(4915 - \sqrt{22088809})/684 \leq |2\nu+1| \leq (4915+\sqrt{22088809})/684$, *then* $\psi(z)$ *is close-to-convex in* $\mathbb{D}$;
4) *for* $\nu \in \mathbb{R}$ *if* $(-7-\sqrt{34})/2 \leq \nu \leq (-7+\sqrt{34})/2$ *the function* $\varphi(z)$ *is close-to-convex in* $\mathbb{D}$, *and if* $(-11-\sqrt{94})/2 \leq \nu \leq (-11+\sqrt{94})/2$ *the function* $\psi(z)$ *is close-to-convex in* $\mathbb{D}$;
5) $\ln M_\varphi(r) = (1+o(1))\frac{r}{4}$ *and* $\ln M_\psi(r) = (1+o(1))\frac{r}{4}$ *as* $r \to \infty$, *where*

$$M_f(r) = \max\{|f(z)| : |z| = r\}.$$

In this article we consider the case $\nu = -\frac{1}{2}$. Then from (3) we have

$$w'' - \frac{z^2}{4}w = 0. \tag{4}$$

Let us find the solution of the equation (4) in the form (1). Since

$$\sum_{n=0}^{\infty}(n+1)(n+2)f_{n+2}z^n - \frac{1}{4}\sum_{n=2}^{\infty}f_{n-2}z^n = 0,$$

so $2f_2 = 0$, $6f_3 = 0$ and $4(n+2)(n+1)f_{n+2} = f_{n-2}$ if $n \geq 2$. We can see that for all $n \in \mathbb{N}$ $f_{4n-2} = f_{4n-1} = 0$, and f_{n+4} depends on f_n. Therefore the solution of the equation (4) is of the form

$$w(z) = C_1\alpha(z^4) + C_2 z\beta(z^4).$$

Let $w(z) = \alpha(z^4)$. Then $w'(z) = 4z^3\alpha'(z^4)$, $w''(z) = 12z^2\alpha'(z^4) + 16z^6\alpha''(z^4)$, and, therefore, the equation (4) in this case is of the form $16z^6\alpha''(z^4) + 12z^2\alpha'(z^4) - \frac{z^2}{4}\alpha(z^4) = 0$. After elementary transformations and replacement z^4 on z we will get

$$64z\alpha''(z) + 48\alpha'(z) - \alpha(z) = 0. \tag{5}$$

If we suppose that $w(z) = z\beta(z^4)$, then, like before, we will get

$$64z\beta''(z) + 80\beta'(z) - \beta(z) = 0. \tag{6}$$

We will find a recurrent formula for the coefficients of the function $\alpha(z) = \sum_{n=0}^{\infty}\alpha_n z^n$, which is the solution of the equation (5). Since

$$64\sum_{n=1}^{\infty}\alpha_{n+1}(n+1)nz^n + 48\sum_{n=0}^{\infty}\alpha_{n+1}(n+1)z^n - \sum_{n=0}^{\infty}\alpha_n z^n = 0,$$

we equate coefficients at the same powers of the variable z and get $48\alpha_1 - \alpha_0 = 0$ and $(64n(n+1) + 48(n+1))\alpha_{n+1} - \alpha_n = 0$ if $n \geq 1$. Note, if $\alpha_0 = 0$ then $\alpha(z) \equiv 0$. Thus we put $\alpha_0 = 1$. Then

$$\alpha_n = \frac{\alpha_{n-1}}{16n(4n-1)}, \quad n \geq 1. \tag{7}$$

For the coefficients of the function $\beta(z) = \sum_{n=0}^{\infty}\beta_n z^n$ which is the solution of the equation (6), we have $80\beta_1 - \beta_0 = 0$ and $16(n+1)(4n+5)\beta_{n+1} - \beta_n = 0$ if $n \geq 1$. If we put $\beta_0 = 1$ then

$$\beta_n = \frac{\beta_{n-1}}{16n(4n+1)}, \quad n \geq 1. \tag{8}$$

1 l-INDEX BOUNDEDNESS

Now we consider l-index boundedness of the functions $\alpha(z)$ and $\beta(z)$. For this purpose we use the following lemma.

Lemma 1 ([4]). *If a function (1) is an analytic in the closed disc* $\overline{\mathbb{D}}_R = \{z : |z| \leq R\}$, $f_0 = 1$, *and*

$$\sum_{n=1}^{\infty}|f_n|R^n \leq a(R) < 1, \tag{9}$$

then $N(f, l;\ \mathbb{D}_R) \leq 1$ *with* $l(|z|) = \frac{1+a(R)}{(1-a(R))(R-|z|)}$.

If $z \in \mathbb{D}_{\xi R}$, $0 < \xi < 1$, then $R - |z| \geq (1-\xi)R$ and Lemma 1 implies $N(f, l;\ \mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \frac{1+a(R)}{(1-\xi)R(1-a(R))}$, because if $N(f, l_*; G) \leq N$ and $l_*(r) \leq l^*(r)$, it is easy to prove [3, p.23], that $N(f, l^*; G) \leq N$. Therefore the next lemma is true.

Lemma 2. *If an entire function (1) satisfies (9) and* $f_0 = 1$, *then for every* $\xi \in (0,1)$ *and* $R \in (0, +\infty)$ *the inequality* $N(f, l;\ \mathbb{D}_{\xi R}) \leq 1$ *holds with* $l(|z|) \equiv \frac{1+a(R)}{(1-\xi)R(1-a(R))}$.

Using (7) we have

$$\sum_{k=1}^{\infty}|\alpha_k|R^k = \frac{R}{48} + \sum_{k=2}^{\infty}|\alpha_k|R^k = \frac{R}{48} + \sum_{k=2}^{\infty}\frac{|\alpha_{k-1}|R^k}{16k(4k-1)} = \frac{R}{48} + \sum_{k=1}^{\infty}\frac{R}{16(k+1)(4k+3)}|\alpha_k|R^k.$$

That is

$$\sum_{k=1}^{\infty}\left(1 - \frac{R}{16(k+1)(4k+3)}\right)|\alpha_k|R^k = \frac{R}{48}.$$

Since $\frac{R}{16(k+1)(4k+3)} \leq \frac{R}{224}$, if $R < 224$, then above equality implies

$$\sum_{k=1}^{\infty}|\alpha_k|R^k \leq \frac{R/48}{1-(R/224)} = \frac{14R}{672-3R} = a(R).$$

Therefore, to use Lemma 2 it is necessary $\frac{R}{48} + \frac{R}{224} < 1$. That is $R < \frac{672}{17}$. For such R by Lemma 2 we have $N(\alpha, l;\ \mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \frac{672+11R}{(1-\xi)R(672-17R)}$.

Now we consider l-index boundedness of the function $\alpha(z)$ in $\mathbb{C}\setminus\mathbb{D}_{\xi R}$. For this purpose we use the fact that $\alpha(z)$ satisfies differential equation (5), and therefore we have $\alpha''(z) = -\frac{3}{4z}\alpha'(z) + \frac{1}{64z}\alpha(z)$. If $|z| \geq \xi R$, $l = 1/(\xi R)$ and $R < 672/17$, then we obtain

$$\frac{|\alpha''(z)|}{2!l^2} \leq \frac{3}{8}\frac{|\alpha'(z)|}{1!l} + \frac{\xi R}{128}|\alpha(z)| \leq \max\left\{\frac{|\alpha'(z)|}{1!l}, |\alpha(z)|\right\}. \qquad (10)$$

Let us differentiate the equation (5) n times. Then we obtain

$$64z\alpha^{(n+2)}(z) + (64n+48)\alpha^{(n+1)}(z) - \alpha^{(n)}(z) = 0.$$

Thus, if $|z| \geq \xi R$, $l = 1/(\xi R)$ and $R < 672/17$, then for all $n \geq 1$ we get

$$\frac{|\alpha^{(n+2)}(z)|}{(n+2)!l^{n+2}} \leq \frac{64n+48}{64(n+2)|z|l}\frac{|\alpha^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{1}{64(n+2)(n+1)|z|l^2}\frac{|\alpha^{(n)}(z)|}{n!l^n}$$
$$\leq \frac{64n+48}{64(n+2)}\frac{|\alpha^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{\xi R/(n+1)}{64(n+2)}\frac{|\alpha^{(n)}(z)|}{n!l^n} \leq \max\left\{\frac{|\alpha^{(n+1)}(z)|}{(n+1)!l^{n+1}}, \frac{|\alpha^{(n)}(z)|}{n!l^n}\right\}. \qquad (11)$$

Inequalities (10) and (11) imply for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}\setminus\mathbb{D}_{\xi R}$

$$\frac{|\alpha^{(n)}(z)|}{n!l^n} \leq \max\left\{\frac{|\alpha'(z)|}{1!l}, |\alpha(z)|\right\},$$

that is, $N(\alpha, l; \mathbb{C}\setminus\mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \frac{1}{\xi R}$. Therefore, for all $\xi \in (0, 1)$ and $R \in \left(0, \frac{672}{17}\right)$ inequality $N(\alpha, l) \leq 1$ holds with $l(|z|) \equiv \max\left\{\frac{1}{\xi R}, \frac{672+11R}{(1-\xi)R(672-17R)}\right\}$.

If we put $\xi = \frac{672-17R}{1344-6R}$, then $\frac{1}{\xi R} = \frac{672+11R}{(1-\xi)R(672-17R)} = \frac{1344-6R}{R(672-17R)}$. Therefore for all $R \in (0, 672/17)$ we have $N(\alpha, l) \leq 1$ with $l(|z|) \equiv \frac{1344-6R}{R(672-17R)}$. The minimal value of the last function on $(0, 672/17)$ is $\frac{31+2\sqrt{238}}{336}$ if $R = 224\left(1-\sqrt{\frac{14}{17}}\right)$.

For the function $\beta(z)$ using recurrent formulas (8) we have

$$\sum_{k=1}^{\infty}|\beta_k|R^k = \frac{R}{80} + \sum_{k=2}^{\infty}|\beta_k|R^k = \frac{R}{80} + \sum_{k=2}^{\infty}\frac{|\beta_{k-1}|R^k}{16k(4k+1)} = \frac{R}{80} + \sum_{k=1}^{\infty}\frac{R}{16(k+1)(4k+5)}|\beta_k|R^k.$$

That is

$$\sum_{k=1}^{\infty}\left(1-\frac{R}{16(k+1)(4k+5)}\right)|\beta_k|R^k = \frac{R}{80}. \qquad (12)$$

Since $\frac{R}{16(k+1)(4k+5)} \leq \frac{R}{288}$, so if $R < 288$, then from (12) we get

$$\sum_{k=1}^{\infty}|\beta_k|R^k \leq \frac{R/80}{1-(R/288)} = \frac{18R}{1440-5R}.$$

Therefore to use lemma 2 it is necessary $R < \frac{1440}{23}$. For such R by lemma 2 we obtain $N(\beta, l; \mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \frac{1440+13R}{(1-\xi)R(1440-23R)}$.

To investigate l-index boundedness of the function $\beta(z)$ in $\mathbb{C}\setminus\mathbb{D}_{\xi R}$ we use the fact that $\beta(z)$ is a solution of the equation (6), i.e. $\beta''(z) = -\frac{5}{4z}\beta'(z) + \frac{1}{64z}\beta(z)$. If $|z| \geq \xi R$, $R < 48$ and $l = 1/(\xi R)$ we have

$$\frac{|\beta''(z)|}{2!l^2} \leq \frac{5}{8}\frac{|\beta'(z)|}{1!l} + \frac{\xi R}{128}|\beta(z)| \leq \max\left\{\frac{|\beta'(z)|}{1!l}, |\beta(z)|\right\}. \qquad (13)$$

Let us differentiate the equation (6) n times. Then we obtain

$$64z\beta^{(n+2)}(z) + (64n+80)\beta^{(n+1)}(z) - \beta^{(n)}(z) = 0.$$

Therefore, if $|z| \geq \xi R$, $R < 48$ and $l = 1/(\xi R)$, then for all $n \in \mathbb{N}$ we get

$$\frac{|\beta^{(n+2)}(z)|}{(n+2)!l^{n+2}} \leq \frac{64n+80}{64(n+2)|z|l}\frac{|\beta^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{1}{64(n+2)(n+1)|z|l^2}\frac{|\beta^{(n)}(z)|}{n!l^n}$$
$$\leq \frac{64n+80}{64(n+2)}\frac{|\beta^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{\xi R/(n+1)}{64(n+2)}\frac{|\beta^{(n)}(z)|}{n!l^n} \leq \max\left\{\frac{|\beta^{(n+1)}(z)|}{(n+1)!l^{n+1}}, \frac{|\beta^{(n)}(z)|}{n!l^n}\right\}. \qquad (14)$$

Inequalities (13) and (14) imply that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}\setminus\mathbb{D}_{\xi R}$ inequality

$$\frac{|\beta^{(n)}(z)|}{n!l^n} \leq \max\left\{\frac{|\beta'(z)|}{1!l}, |\beta(z)|\right\}$$

holds, that is $N(\beta, l; \mathbb{C}\setminus\mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \frac{1}{\xi R}$. Therefore, for all $\xi \in (0, 1)$ and $R \in (0, 48)$ inequality $N(\beta, l) \leq 1$ holds with $l(|z|) \equiv \max\left\{\frac{1}{\xi R}, \frac{1440+13R}{(1-\xi)R(1440-23R)}\right\}$.

If we put $\xi = \frac{1440-23R}{2880-10R}$, then $\frac{1}{\xi R} = \frac{1440+13R}{(1-\xi)R(1440-23R)} = \frac{2880-10R}{R(1440-23R)}$. Therefore, for all $R \in (0, 48)$ we have $N(\beta, l) \leq 1$ with $l(|z|) \equiv \frac{2880-10R}{R(1440-23R)}$. The minimal value of the last function on $(0, 48)$ is $\frac{41+2\sqrt{414}}{720}$ if $R = 288\left(1-\sqrt{\frac{18}{23}}\right)$.

Therefore, the following proposition is true.

Proposition 1. *$N(\alpha, l) \leq 1$ with $l(|z|) \equiv \frac{31+2\sqrt{238}}{336}$ and $N(\beta, l) \leq 1$ with $l(|z|) \equiv \frac{41+2\sqrt{414}}{720}$.*

2 GEOMETRICAL PROPERTIES

We use following lemma to investigate convexity of the functions $\alpha(z)$ and $\beta(z)$.

Lemma 3 ([2]). *If $\sum_{n=2}^{+\infty} n^2|f_n| \leq |f_1|$, then function (1) is convex in $\mathbb{D}$.*

Using recurrent formula (7) we get

$$\sum_{n=2}^{+\infty} n^2|\alpha_n| \le 4|\alpha_2| + \sum_{n=3}^{+\infty} n^2 \frac{|\alpha_{n-1}|}{16n(4n-1)} = \frac{4}{10752} + \sum_{n=2}^{+\infty} \frac{n+1}{16n^2(4n+3)} n^2|\alpha_n|,$$

that is

$$\sum_{n=2}^{+\infty} \left(1 - \frac{n+1}{16n^2(4n+3)}\right) n^2|\alpha_n| \le \frac{1}{2688}. \tag{15}$$

Since for $n \ge 2$ we have the inequality $1 - \frac{n+1}{16n^2(4n+3)} \ge 1 - \frac{3}{704}$, so (15) implies

$$\sum_{n=2}^{+\infty} n^2|\alpha_n| \le \frac{1/2688}{701/704} < \frac{1}{48} = |\alpha_1|.$$

Applying a similar reasoning to the function $\beta(z)$ we obtain

$$\sum_{n=2}^{+\infty} n^2|\beta_n| \le 4|\beta_2| + \sum_{n=3}^{+\infty} n^2 \frac{|\beta_{n-1}|}{16n(4n+1)} = \frac{4}{23040} + \sum_{n=2}^{+\infty} \frac{n+1}{16n^2(4n+5)} n^2|\beta_n|,$$

that is

$$\sum_{n=2}^{+\infty} \left(1 - \frac{n+1}{16n^2(4n+5)}\right) n^2|\beta_n| \le \frac{1}{5760}.$$

Since $1 - \frac{n+1}{16n^2(4n+5)} \ge 1 - \frac{3}{832}$, so

$$\sum_{n=2}^{+\infty} n^2|\beta_n| \le \frac{1/5760}{829/832} \le \frac{1}{80} = |\beta_1|.$$

Therefore, the next proposition is true.

Proposition 2. *Functions $\alpha(z)$ and $\beta(z)$ are convex in $\mathbb{D}$.*

3 Growth

The next proposition describes the growth of the functions $\alpha(z)$ and $\beta(z)$.

Proposition 3. $\ln M_\alpha(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *and* $\ln M_\beta(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *as* $r \to \infty$, *where* $M_f(r) = \max\{|f(z)| : |z| = r\}$.

Really, since

$$\alpha_n = \frac{\alpha_{n-1}}{16n(4n-1)} = \frac{\alpha_0}{16^n n!} \prod_{k=1}^{n} \frac{1}{4k-1} = \frac{1}{64^n (n!)^2} \prod_{k=1}^{n} \left(1 + \frac{1}{4k-1}\right),$$

so for every $\varepsilon > 0$ and for all $n \in \mathbb{N}$

$$\frac{1}{64^n (n!)^2} \le \alpha_n \le \frac{K(1+\varepsilon)^n}{64^n (n!)^2}, \tag{16}$$

where $K = K(\varepsilon)$ is a positive constant.

To obtain an asymptotic behavior of the function $\alpha(z)$ from inequality (16) we will consider the function $g(r) = \sum_{n=0}^{+\infty} \frac{r^n}{(n!)^2}$, where $r \ge 0$. Let $\mu_g(r) = \max\{r^n/(n!)^2 : n \ge 0\}$ be the maximal term of the last series and $\nu_g(r) = \max\{n : r^n/(n!)^2 = \mu_g(r)\}$ be the central index. Then $\nu_g(r) = n$ for $n^2 \le r < (n+1)^2$, therefore $\nu_g(r) = (1+o(1))\sqrt{r}$ if $r \to +\infty$. Therefore

$$\ln \mu_g(r) = \ln \mu_g(1) + \int_1^r \frac{\nu_g(t)}{t} dt = (1+o(1))2\sqrt{r}, \quad r \to +\infty,$$

and by the Borel's theorem we get $\ln M_g(r) = (1+o(1)) \ln \mu_g(r) = (1+o(1))2\sqrt{r}$, $r \to +\infty$. From (16), in view of arbitrariness of ε, we have

$$\ln M_\alpha(r) = (1+o(1))2\sqrt{\frac{r}{64}} = (1+o(1))\frac{\sqrt{r}}{4}, \quad r \to +\infty.$$

Similar we get asymptotical equality $\ln M_\beta(r) = (1+o(1))\frac{\sqrt{r}}{4}$, $r \to +\infty$.

4 Main theorem

Propositions 1–3 imply the following theorem.

Theorem 1. *The general solution of* (4) *can be written in the form* $w(z) = C_1\alpha(z^4) + C_2 z\beta(z^4)$, *where entire functions* $\alpha(z)$ *and* $\beta(z)$ *are convex in* $\mathbb{D}$, $N(\alpha, l) \le 1$ *with* $l(|z|) \equiv \frac{31+2\sqrt{238}}{336}$ *and* $N(\beta, l) \le 1$ *with* $l(|z|) \equiv \frac{41+2\sqrt{414}}{720}$, *also* $\ln M_\alpha(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *and* $\ln M_\beta(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *as* $r \to \infty$, *where* $M_f(r) = \max\{|f(z)| : |z| = r\}$.

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Досліджено властивості функцій $\alpha(z)$ та $\beta(z)$ таких, що $\alpha(z^4)$ та $z\beta(z^4)$ є лінійно незалежними розв'язками рівняння Вебера $w'' - (\frac{z^2}{4} - \nu - \frac{1}{2})w = 0$ при $\nu = -\frac{1}{2}$, а саме обмеженість l-індексу, опуклість та можливе зростання.

Ключові слова і фрази: ціла функція, обмеженість l-індексу, зростання, опукла функція, рівняння Вебера.

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STRICTLY DIAGONAL HOLOMORPHIC FUNCTIONS ON BANACH SPACES

In this paper we investigate the boundedness of holomorphic functionals on a Banach space with a normalized basis $\{e_n\}$ which have very special form $f(x) = f(0) + \sum_{n=1}^{\infty} c_n x_n^n$ and which we call strictly diagonal. We consider under which conditions strictly diagonal functions are entire and uniformly continuous on every ball of a fixed radius.

Key words and phrases: holomorphic functions on Banach space, base on Banach space.

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INTRODUCTION AND PRELIMINARIES

Let X be a separable complex Banach space with a normalized basis $\{e_n\}_{n=1}^{\infty}$. A holomorphic function f on an open ball $B(0,r)$ of X centered at zero (of finite or infinite radius r) will be called *strictly diagonal* with respect to the basis if it is of the form

$$f(x) = f(0) + \sum_{n=1}^{\infty} c_n x_n^n, \qquad x \in X, \qquad \text{where} \qquad x = \sum_{n=1}^{\infty} x_n e_n. \tag{1}$$

We can *associate* a formal power series with f in such way

$$\gamma(t) = \sum_{n=0}^{\infty} c_n t^n, \qquad c_0 = f(0), \qquad t \in \mathbb{C}$$

and we will write $\gamma = \gamma_f$ and $f = f_\gamma$ if it is necessary. Note that the strictly diagonal function $f(x) = \sum_{n=1}^{\infty} x_n^n$ is the well-known example [4, p. 169] of entire function on ℓ_p, $1 \le p < \infty$ or on c_0 which is not of bounded type (the radius of boundedness at zero is equal to one). On the other hand its associated series $\gamma(t)$ well defines a holomorphic function only on the open unit disk $\mathbb{D}_1 \subset \mathbb{C}$. More examples of entire holomorphic functions which are not bounded on all bounded sets can be found in [1, 2, 3].

The purpose of this paper is to examine properties of strictly diagonal holomorphic functions in terms of associated power series and construct some new interesting examples of holomorphic functions on X.

Le us recall that a continuous function $f\colon X \to \mathbb{C}$ is said to be *holomorphic* at a point $a \in X$ if it has power series representation

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

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in a neighborhood of a, where f_n are continuous n-homogeneous polynomials. A function f is *entire* if it is holomorphic at each point of X. The space of all entire functions on X is denoted by $H(X)$.

The *radius of uniform convergence* of a function f at a can be calculated by formula

$$\rho_a(f) = (\limsup_{n\to\infty} \|f_n\|^{\frac{1}{n}})^{-1}$$

and coincides with the radius of boundedness. In particular, each entire function is uniformly bounded on the ball $B(a,r)$ centered at a of radius r if $r < \rho_a(f)$ and unbounded on $B(a,r)$ if $r > \rho_a(f)$.

For details on holomorphic functions on Banach spaces we refer the reader to [4, 5, 7].

1 MAIN RESULTS

Throughout in this section f is a strictly diagonal function defined by (1).

Theorem 1. *Let $\delta > 0$ and*

$$\gamma(t) = \sum_{n=0}^{\infty} c_n t^n$$

converges in the open δ-disk $\mathbb{D}_\delta = \{t \in \mathbb{C}\colon |t| < \delta\}$. Then $f_\gamma \in H(X)$ and $\rho_z(f_\gamma) \ge \delta$ for every $z \in X$.

Proof. For a given $x \in X$ let n_0 be a number such that $|x_n| \le r < \delta$ for every $n > n_0$. Then

$$|f_\gamma(x)| \le \Big|\sum_{k=0}^{n_0} c_k x_k\Big| + \sum_{k=n_0+1}^{\infty} |c_k||x_k| \le \Big|\sum_{k=0}^{n_0} c_k x_k\Big| + \sum_{k=n_0+1}^{\infty} |c_k| r^k < \infty.$$

So f_γ is well-defined at any point of X. Clearly f_γ is G-holomorphic and

$$\rho_0(f_\gamma) = \big(\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}\big)^{-1} = \rho_0(\gamma) \ge \delta.$$

This, in particular, means that f_γ is locally bounded at 0 and so it is holomorphic. Let z be a fixed element in X. For any $0 < r < \delta$ let m_0 be a number such that $|z_n| < \frac{\delta - r}{2}$ $\forall n > m_0$. Then for every $x \in X$, $\|x\| < r$, we have

$$|f_\gamma(x+z)| \le \Big|\sum_{k=0}^{m_0} c_k (z_k + x_k)^k\Big| + \sum_{m_0+1}^{\infty} c_k |(z_k + x_k)|^k \le \Big|\sum_{k=0}^{m_0} c_k (z_k + x_k)^k\Big| + \Big|\gamma\Big(\frac{\delta - r}{2} + r\Big)\Big|.$$

Let us denote

$$c(z,r) := \Big|\sum_{k=0}^{m_0} c_k (z_k + x_k)^k\Big| + \Big|\gamma\Big(\frac{\delta - r}{2} + r\Big)\Big|.$$

Then for every $z \in X$ and $r < \delta$, f_γ is bounded in $B(z,r)$ by the constant $c(z,r)$ which depends only on z and r. That is, $\rho_z(f_\gamma) \ge \delta$. □

Definition 1.1. *A basis $\{e_n\}_{n=1}^{\infty}$ is said to be boundedly complete if for every sequence of numbers $\{b_n\}_{n=1}^{\infty}$ such that $\sup \|\sum_{n=1}^{m} b_n e_n\| < \infty$ the series $\sum_{n=1}^{\infty} b_n e_n$ converges to a vector in X.*

Note that the standard basis in ℓ_p, $1 \le p < \infty$, is boundedly complete while in c_0 it is not. Moreover if $\{e_n\}_{n=1}^{\infty}$ is not boundedly complete, then it contains a subsequence equivalent to the standard basis in c_0 (see [6]).

Definition 1.2. *We say that K is the index of boundedness of $\{e_n\}_{n=1}^{\infty}$ if*

$$\|x\| = \|\sum_{n=1}^{\infty} x_n e_n\| = \delta > 0$$

implies that the cardinality of set $\{x_k\colon |x_k| = \delta\}$ does not exceed K.

Theorem 2. *Let $\{e_n\}_{n=1}^{\infty}$ be a normalized basis of a Banach space X which has a finite index of boundedness K and $\gamma(t) = \sum_{n=0}^{\infty} c_n t^n$ is holomorphic and bounded on the disk $\mathbb{D}_\delta$. Then $f_\gamma \in H(X)$ and for every $z \in X$, f_γ is bounded on $B(z,\delta)$.*

Proof. From Theorem 1 it follows that $f_\gamma \in H(X)$. For a given $x \in X$, $\|x\| < 1$, we have

$$|f_\gamma(x)| \le \sum_{n=0}^{K} c_n \delta^n + \sup_{|t|<\delta} |\gamma(t)|.$$

So

$$\sup_{\|x\|<1} |f_\gamma(x)| \le \sum_{n=0}^{K} c_n \delta^n + \sup_{|t|<\delta} |\gamma(t)| < \infty$$

and f_γ is bounded on $B(0,\delta)$. Using the same work like in Theorem 1 we can show that f_γ is bounded on $B(z,\delta)$ for every fixed $z \in X$. □

Definition 1.3. *Let us suppose that there are $0 < \varepsilon < 1$ and positive integer K_ε such that $\|x\| = 1$ implies $\operatorname{card}\{x_n\colon |x_n| \le 1-\varepsilon\} \le K_\varepsilon < \infty$. Then we say that K_ε is the index of ε-boundedness of the basis $\{e_n\}_{n=1}^{\infty}$.*

Clearly that if X has an index of ε-boundedness K_ε for some $\varepsilon > 0$, then $K_\varepsilon = K$.

Example 1. *Let $X = \bigoplus_{k=1}^{\infty} \ell_\infty^k$ (the ℓ_1-sum). That is, for every*

$$x = \sum_{k=1}^{\infty}\sum_{j=1}^{k} x_j^k e_j^k = (x_1^1, x_1^2, x_2^2, x_1^3, x_2^3, x_3^3, \ldots), \qquad x \in X,$$

we have

$$\|x\| = \sum_{k=1}^{\infty} \max_{1\le j\le k} |x_j^k|.$$

Basis $\{e_j^k\}_{k=1,j=1}^{\infty,\ k}$ is boundedly complete. Indeed, let $\{b_j^k\}_{k=1,j=1}^{\infty,\ k}$ be a sequence of numbers such that $\sum_{k=1}^{m} \max_{1\le j\le k} |b_j^k| < c$ for every m and some $c > 0$. Then $\sum_{k=1}^{\infty} \max_{1\le j\le k} |b_j^k|$ converges and so $\sum b_j^k e_j^k \in X$. On the other hand for every $K \in \mathbb{N}$ we can pick

$$x_0 = e_1^{k+1} + \cdots + e_{k+1}^{k+1}$$

with $\|x_0\| = 1$ and so $\{e_j^k\}_{k=1,j=1}^{\infty,\ k}$ has no finite index of boundedness.

Example 2. *Let X be the ℓ_1-sum of ℓ_n, $X = \bigoplus_{n=1}^{\infty} \ell_n^n$ and $\{e_j^k\}_{k=1,j=1}^{\infty,\ k}$ be the natural basis. This basis has the index of boundedness $K = 1$. Indeed, suppose $\|x\| = 1$ and for two different coordinates $|x_j^k| = 1$ and $|x_i^s| = 1$. We have two cases:*

1) if $k = s$, then $\|x\| \ge (|x_j^k|^k + |x_i^k|^k)^{\frac{1}{k}} > 1$,

2) if $k \ne s$, then $\|x\| \ge 2$.

This contradicts our assumption. So, just one coordinate may have the absolute value equals one.

Let $0 < \varepsilon < 1$ and K_ε be a fixed positive integer. Let us find $k_0 \in \mathbb{N}$ such that $(1-\varepsilon)^{k_0} < \frac{1}{K_\varepsilon+1}$. Let $m \ge 2\max(k_0, K_\varepsilon+1)$ and

$$x_0 = (1-\varepsilon)e_1^m + \ldots + (1-\varepsilon)e_{K_\varepsilon+1}^m,$$

then

$$\begin{aligned}\|x_0\|^m &= (K_\varepsilon+1)(1-\varepsilon)^m = (K_\varepsilon+1)(1-\varepsilon)^{\frac{2m}{2}}\\ &< (K_\varepsilon+1)(1-\varepsilon)^{\frac{m}{2}}(1-\varepsilon)^{\frac{m}{2}} < (K_\varepsilon+1)\frac{1}{(K_\varepsilon+1)}(1-\varepsilon)^{\frac{m}{2}},\end{aligned}$$

that is,

$$\|x_0\| \le ((1-\varepsilon)^{\frac{m}{2}})^{\frac{1}{m}} = (1-\varepsilon)^{\frac{1}{2}}.$$

It means that the index of ε-boundedness of the basis is greater than K_ε. Since K_ε is arbitrary, the basis has no finite index of ε-boundedness.

Theorem 3. *Let $\{e_n\}_{n=1}^{\infty}$ be a basis of a Banach space X which has an index of ε-boundedness K_ε for every $0 < \varepsilon < 1$ and $\gamma(t) = \sum_{n=0}^{\infty} c_n t^n$ converges in the disk $\mathbb{D}_1$. Then f_γ is uniformly continuous on $B(z,1)$ for every $z \in X$.*

Proof. Let us prove the statement for the case $B(0,1)$. The general case follows from there like in Theorem 1. Note that $\gamma(t)$ is uniformly continuous on the closed disk $\overline{\mathbb{D}}_\rho$ for every $0 < \rho < 1$. For a given $0 < \varepsilon < 1$ let $\omega > 0$ be such that

$$|\gamma(t_1) - \gamma(t_2)| < \varepsilon \tag{2}$$

if only $|t_1 - t_2| < \delta$ for $t_1, t_2 \in \overline{\mathbb{D}}_{1-\varepsilon/2}$. Let $x, y \in X$, $\|x\| \le 1$, $\|y\| \le 1$,

$$x = \sum_{n=1}^{\infty} x_n e_n, \quad y = \sum_{n=1}^{\infty} y_n e_n.$$

Then there is a number $m \le K_\varepsilon + 1$ such that for

$$\widetilde{x} = \sum_{n=m}^{\infty} x_n e_n \quad \text{and} \quad \widetilde{y} = \sum_{n=m}^{\infty} y_n e_n$$

$\|\widetilde{x}\| < 1-\varepsilon$ and $\|\widetilde{y}\| < 1-\varepsilon$. Clearly that f_γ is uniformly continuous on $B(0,1)$ if and only if

$$f_\gamma^c := \sum_{n=m}^{\infty} c_n x_n^n$$

is uniformly continuous on $B(0,1)$. If $\|\widetilde{x} - \widetilde{y}\| < \delta$, then $\|x_k - y_k\| < \delta$ for $k \ge m$. Let $r = \sup_{k\ge m} \|x_k - y_k\|$. Then from (2) we obtain

$$\|f_\gamma(x) - f_\gamma(y)\| = |\sum_{n=m}^{\infty} c_n(x_n^n - y_n^n)| \le \sum_{n=m}^{\infty} |c_n(x_n^n - y_n^n)| \le \sum_{n=m}^{\infty} c_n r^n < \varepsilon.$$

□

Example 3. *Let $\gamma(t) = \sum_{n=1}^{\infty} t^n$, then the entire function f_γ is uniformly continuous on a unit ball centered at any point in ℓ_p, $1 \leq p < \infty$. But it is not bounded in the unit ball in c_0. Indeed let $x^n = e_1 + e_2 + \ldots + e_n \in c_0$, then $f(x^n) = n \to \infty$. By the same way it is possible to show that if $\gamma(t)$ is unbounded in $\mathbb{D}_1 \subset \mathbb{C}$, then $f_\gamma(x)$ is unbounded in the unit ball of c_0.*

Proposition 1.1. *f_γ is bounded on $B(z,r) \subset c_0$ for every $z \in c_0$ if and only if $\gamma(t)$ converges absolutely on $\overline{\mathbb{D}}_r$.*

Proof. If $\gamma(t)$ converges absolutely on $\overline{\mathbb{D}}_r$, then it is easy that f_γ is bounded on $B(z,r) \subset c_0$ for every $z \in c_0$. To prove the converse statement without loss of the generality we assume that $r = 1$. If $\gamma(t) = \sum_{n=1}^{\infty} c_n t^n$ does not converges absolutely on $\overline{\mathbb{D}}_1$, then there are numbers b_n, $|b_n| = 1$, such that $\sum_{n=1}^{\infty} c_n b_n \to \infty$ as $n \to \infty$.

Let $x_n = \sqrt[n]{b_n}$ and $x^n = \sum_{n=1}^{m} x_n e_n$. Clearly $\|x^m\|_{c_0} = 1$ and $f_\gamma(x^m) = \sum_{n=1}^{m} c_n b_n = m \to \infty$ so $f_\gamma(x)$ is unbounded on $B(0,1)$. □

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Федак О.І., Загороднюк А.В. *Строго діагональні голоморфні функції на банахових просторах* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 254–258.

Досліджено обмеженість голоморфних функцій на банахових просторах з базисом $\{e_n\}$, які мають дуже спеціальний вигляд $f(x) = f(0) + \sum_{n=1}^{\infty} c_n x_n^n$ і які ми називаємо строго діагональними. Розглянуто при яких умовах строго діагональні функції будуть цілими і рівномірно обмеженими на всіх кулях фіксованого радіуса.

Ключові слова і фрази: голоморфні функції на банахових просторах, базиси в банахових просторах.

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HAJJEJ Z., BALEGH M.

UNIFORM BOUNDARY CONTROLLABILITY OF A DISCRETE 1-D SCHRÖDINGER EQUATION

In this paper we study the controllability of a finite dimensional system obtained by discretizing in space and time the linear 1-D Schrödinger equation with a boundary control. As for other problems, we can expect that the uniform controllability does not hold in general due to high frequency spurious modes. Based on a uniform boundary observability estimate for filtered solutions of the corresponding conservative discrete system, we show the uniform controllability of the projection of the solutions over the space generated by the remaining eigenmodes.

Key words and phrases: observability, controllability, filtering.

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INTRODUCTION

Let us consider the 1-D Schrödinger equation

$$\begin{cases} u_t(x,t) + iu_{xx}(x,t) = 0, & 0 < x < 1, \ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u^0(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $u^0 \in H_0^1(0,1)$. It is well known that the energy

$$E(t) = \frac{1}{2}\int_0^1 |u_x(x,t)|^2 \, dx \tag{2}$$

of the solutions is conserved in time. Applying Fourier series techniques one can prove a boundary observability inequality showing that, for every $T > 0$, there exists $C = C(T) > 0$ such that

$$E(0) \leq C \int_0^T |u_x(1,t)|^2 \, dt \tag{3}$$

for every solution of (1).

As a consequence of this observability inequality and the HUM method [10], the following boundary controllability property may be proved.

For all $T > 0$ and $y^0 \in H^{-1}(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1, \ 0 < t < T, \\ y(0,t) = 0, \ y(1,t) = v, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1, \end{cases} \tag{4}$$

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satisfies $y(T) = 0$.

This article aims at studying the observability and the controllability properties for space-discrete and fully discrete approximations schemes of (1) and (4).

In the last years many works have dealt with the numerical approximations of the control problem of the wave equation using the HUM approach [1, 4, 11]. It is by now well-known that discretization processes may create high frequency spurious solutions which might lead to non-uniform observability properties. The conclusion was that the controllability property is not uniform as the discretization parameter h goes to zero and, consequently, the control of the discrete model do not converge to the control of the continuous model. Some remedies are then necessary to restore the convergence of the discrete control to the continuous one. We can mention the Tychonoff regularization [6], a mixed finite element method [1], or a filtering technique [7]. In the context of fully discrete conservative equations, we refer to [3], which deals with very general approximation schemes for conservative linear systems. For space semi-discrete approximations of Shrödinger equation, we mention the work [2] which study interior observability and controllability properties, based on spectral estimates. Let us also mention that the time semi-discrete Schrödinger equation has been studied in [13]. Our article seems to be the first one that deals with fully discrete Schrödinger equation in details providing an uniform result of boundary controllability.

The outline of this paper is as follows.

The second section briefly recalls some controllability results for the Schrödinger equation. In section 3, we study the space discrete observability and controllability properties. Section 4 is devoted to prove observability and controllability problems of fully discrete approximation schemes of (1) and (4).

1 THE CONTINUOUS PROBLEM

In this section, we recall briefly the controllability property of the Schrödinger system (4) (see [10, 14] for more details).

Theorem 1. *For all $T > 0$ and $(y^0) \in H^{-1}(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of (4) satisfies $y(T) = 0$.*

Multiplying in (4) by $\bar{u}$, integrating by parts in $(0,1) \times (0,T)$ and using the equations (1) that u satisfies we deduce that

$$i \int_0^T v\bar{u}_x(1)dt + \int_0^1 y^0\bar{u}^0 dx = \int_0^1 y(T)\bar{u}(T)dx.$$

Taking imaginary parts in the last equality, we deduce that

$$\mathrm{Re} \int_0^T v\bar{u}_x(1)dt + \mathrm{Im} \int_0^1 y^0\bar{u}^0 dx = 0.$$

Here and in the sequel Re, Im and $\bar{u}$ stand respectively for the real part, the imaginary part of a complex number and the conjugate of u.

The control of minimal L^2-norm can be obtained by minimizing functional $J : H_0^1(0,1) \to \mathbb{R}$ defined as follows:

$$J(u^0) = \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt - \mathrm{Im} \int_0^1 y^0\bar{u}^0 dx. \tag{5}$$

The functional J is continuous and convex. Moreover, J is coercive because of the observability inequality (3). Then, the following result holds.

Theorem 2. *Given any $T > 0$ and $y^0 \in H^{-1}(0,1)$ the functional J has an unique minimizer $\hat{u}^0 \in H_0^1(0,1)$. If $\hat{u}$ is the corresponding solution of (1) with initial data $\hat{u}^0$ then $v(t) = -\hat{u}_x(1,t)$ is the control of (4) with minimal L^2-norm.*

As said in the introduction, the controllability property is equivalent to the observability inequality (3).

Let us finally remark that the solution of (1) admits the Fourier expansion

$$u(x,t) = \sum_{k>0} c_k e^{ik^2\pi^2 t} \sin(k\pi x),$$

with suitable Fourier coefficients depending on the initial data u^0.

2 SPACE SEMI-DISCRETIZATIONS

In this section, we consider the space semi-discrete version of the continuous observability and controllability problems. Let N be a nonnegative integer. Set $h = \frac{1}{N+1}$ and consider the subdivision of $(0,1)$ given by

$$0 = x_0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1,$$

i.e., $x_j = jh$ for all $j = 0, \ldots, N+1$. Consider the following finite difference approximation of (4):

$$\begin{cases} y_j'(t) + i\frac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2} = 0, & 0 < t < T,\ j = 1, \ldots, N, \\ y_0(t) = 0,\ y_{N+1}(t) = v_h(t), & 0 < t < T, \\ y_j(0) = y_j^0, & j = 1, \ldots, N. \end{cases} \tag{6}$$

As in the context of the continuous Schrödinger equation above, we consider the uncontrolled system

$$\begin{cases} u_j'(t) + i\frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} = 0, & 0 < t < T,\ j = 1, \ldots, N, \\ u_0(t) = u_{N+1}(t) = 0, & 0 < t < T, \\ u_j(0) = u_j^0, & j = 1, \ldots, N. \end{cases} \tag{7}$$

The energy of system (7) is given by

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N} \left| \frac{u_{j+1}(t) - u_j(t)}{h} \right|^2,$$

which is a discretization of the continuous energy $E(t)$. It is easy to see that the energy E_h is conserved along time for the solutions of (7), i.e.

$$E_h(t) = E_h(0) \qquad \text{for all} \quad 0 < t < T.$$

We observe that the system (7) can be rewritten in the following simplified form

$$u_h'(t) - iA_h u_h(t) = 0,\ 0 < t < T,\ u_h(0) = u_h^0, \tag{8}$$

where u_h stands for the column vector $(u_1, \ldots, u_N)^T$, A_h denotes the matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

entering in the finite difference discretization of the Laplacian with Dirichlet boundary conditions. We consider the eigenvalue problem associated with (7):

$$\begin{cases} i\frac{\Phi_{j+1}-2\Phi_j+\Phi_{j-1}}{h^2} = \beta_j\Phi_j, & j=1,\ldots,N, \\ \Phi_0 = \Phi_{N+1} = 0. \end{cases} \tag{9}$$

Let us denote by $\beta_{1,h},\ldots,\beta_{N,h}$ the N eigenvalues of (9). These eigenvalues can be computed explicitly [8]. We have

$$\beta_{k,h} = -i\lambda_{k,h} = -i\frac{4}{h^2}\sin^2\left(\frac{\pi hk}{2}\right), \quad k=1,\ldots,N.$$

The eigenfunction $\Phi^{k,h} = (\Phi_1^{k,h},\ldots,\Phi_N^{k,h})$ associated to the eigenvalue $\beta_{k,h}$ can also be computed explicitly:

$$\Phi_j^{k,h} = \sin(j\pi hk), \quad j=1,\ldots,N.$$

Solutions of (7) admit a Fourier development on the basis of eigenvectors of system (9). More precisely, every solution $u_h = (u_j)_j$ of (7) can be written as

$$u_h(t) = \sum_{k=1}^{N} a_k e^{i\lambda_{k,h}t}\Phi^{k,h},$$

for suitable coefficients $a_k \in \mathbb{C}$, $k=1,\ldots,N$, that can be computed explicitly in terms of the initial data.

2.1 Uniform observability of (7)

The main goal of this subsection is to analyze the following discrete version of (3):

$$E_h(0) \le C(T,h)\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt, \tag{10}$$

where $C(T,h)$ is independent of the solution of (7).

The observability inequality (10) is said to be uniform, if the constants $C(T,h)$ are bounded uniformly in h, as $h \to 0$. However, the following result asserts that this is false.

Theorem 3. *Let u is a solution of (7). For any $T>0$ we have*

$$\sup_u \left[\frac{E_h(0)}{\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt}\right] \to \infty \quad as \quad h \to 0.$$

Before getting into the proof of Theorem 3, let us recall the following property of the eigenvectors of (9) proved in [7].

Lemma 1. *For any eigenvector Φ with eigenvalue β of (9), the following identity holds:*

$$h\sum_{j=0}^{N}\left|\frac{\Phi_{j+1}-\Phi_j}{h}\right|^2 = \frac{2}{4-i\beta h^2}\left|\frac{\Phi_N}{h}\right|^2 = \frac{2}{4-\lambda h^2}\left|\frac{\Phi_N}{h}\right|^2.$$

Proof of Theorem 3. For $h>0$, consider the particular solution of (7)

$$u_h(t) = e^{i\lambda_{N,h}t}\Phi^{N,h}.$$

For this solution we have

$$E_h(0) = h\sum_{j=0}^{N}\left|\frac{\Phi_{j+1}^{N,h}-\Phi_j^{N,h}}{h}\right|^2 = \frac{2}{4-\lambda_{N,h}h^2}\left|\frac{\Phi_N^{N,h}}{h}\right|^2.$$

On the other hand,

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = T\left|\frac{\Phi_N^{N,h}}{h}\right|^2.$$

Note that

$$4-\lambda_{N,h}h^2 = 4-4\sin^2\left(\frac{\pi}{2}-\frac{\pi h}{2}\right) = 4-4\cos^2\left(\frac{\pi h}{2}\right) \to 0 \quad \text{as} \quad h\to 0.$$

Thus, the result is established. □

To overcome this obstacle, we rule out the high frequency spurious modes. We define

$$C_s = \text{span}\{\Phi^{k,h} \text{ such that } \lambda_{k,h} \le s\}.$$

In order to obtain a positive counterpart to Theorem 3, we have to introduce suitable subclasses of solutions of (7) generated by eigenvectors of (9) associated with eigenvalues such that $\lambda h^2 \le \gamma$. For a given $\gamma \in (0,4)$, we take solutions of (7) in C_{γ/h^2}.

We are ready to prove the following uniform boundary observability of the discrete Schrödinger equation.

Theorem 4. *Let $0<\gamma<4$. For all $T>0$ there exist $C=C(T,\gamma)>0$ such that*

$$E_h(0) \le C\int_0^T \left|\frac{u_N(t)}{h}\right|^2$$

for every solution u_h of (8) with $u_h^0 \in C_{\gamma/h^2}$.

Sketch of the proof. In the range of eigenvalues $\lambda h^2 \le \gamma$, according to the identity of Lemma 1, it follows that

$$h\sum_{j=0}^{N}\left|\frac{\Phi_{j+1}-\Phi_j}{h}\right|^2 \le \frac{2}{4-\gamma}\left|\frac{\Phi_N}{h}\right|^2 \tag{11}$$

for any eigenvector $\Phi = (\Phi_1,\ldots,\ \Phi_N)$ associated to an eigenvalue β such that $i\beta h^2 \le \gamma$, or equivalent $\lambda h^2 \le \gamma$.

Let us now consider a solution u_h of (7) in the class C_{γ/h^2}. It can be written as

$$u_h(t) = \sum_{\lambda_{k,h}h^2\le\gamma} a_k e^{i\lambda_{k,h}t}\Phi^{k,h}.$$

As was proved in [9], roughly speaking, the asymptotic gap tends to infinity as $k\to\infty$, uniformly on the parameter h. Then applying Lemma 2.3 [9] and using (11) we deduce that for $T>0$,

$$C(T,\gamma)\int_0^T \left|\frac{u_N(t)}{h}\right|^2 \ge \sum_{\lambda_{k,h}h^2\le\gamma}|a_k|^2 h\sum_{j=0}^{N}\left|\frac{\Phi_{j+1}^{k,h}-\Phi_j^{k,h}}{h}\right|^2.$$

Moreover,

$$E_h(0) = \frac{1}{2}\sum_{\lambda_{k,h}h^2\le\gamma}|a_k|^2 h\sum_{j=0}^{N}\left|\frac{\Phi_{j+1}^{k,h}-\Phi_j^{k,h}}{h}\right|^2.$$

Therefore, we obtain the desired inequality. □

2.2 Uniform controllability of (6)

In this subsection we apply the observability result obtained above to analyze the controllability properties of the semi-discrete system (6).

For every $s \in \mathbb{R}$, introduce the finite dimensional Hilbert spaces

$$H_h^s = span\{\Phi^{1,h}, \dots, \Phi^{N,h}\}$$

endowed with the norm

$$\|f_h\|^2_{H_h^s} = \sum_{k=1}^{N} \lambda^s_{k,h} |d_k|^2, \quad \text{whenever} \quad f_h = \sum_{k=1}^{N} d_k \Phi^{k,h},$$

where $\lambda_{k,h} = \frac{4}{h^2}\sin^2(\frac{k\pi h}{2})$.

Let $0 < \gamma < 4$ and $T > 0$. The partial controllability problem of system (6) in the space H_h^{-1} consists in finding a control $v_h \in L^2(0,T)$ such that the solution $y_h = (y_j)_j$ of (6) satisfies

$$\Pi_\gamma(y_h(T)) = 0, \tag{12}$$

where Π_γ is the orthogonal projection over C_{γ/h^2}.

Multiplying (6) by $\bar{u}_j$, adding in j and integrating in time, we get

$$\text{Im } h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 - \text{Re} \int_0^T v_h(t) \frac{\bar{u}_N(t)}{h} dt = 0.$$

We obtain the following characterization of the partial controllability property of system (6).

Lemma 2. *Let $T > 0$ and $0 < \gamma < 4$. Problem (6) is partially controllable in H_h^{-1} if for every $y_h^0 \in H_h^{-1}$ there exists a control v_h such that*

$$Im\ h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 = Re \int_0^T v_h(t) \frac{\bar{u}_N(t)}{h} dt,$$

for any initial data $u_h^0 \in C_{\gamma/h^2}$.

The following uniform partial controllability property holds in the space C_{γ/h^2}.

Theorem 5. *For all $T > 0$ and $0 < \gamma < 4$, the problem (6) is partially controllable in H_h^{-1} for all $0 < h < 1$. Moreover, we have:*

(a) the corresponding controls v_h in the semi-discrete system (6) satisfying (12) are bounded in $L^2(0,T)$;

(b) the controls v_h converge as $h \to 0$ to a control $v \in L^2(0,T)$ of the minimal $L^2(0,T)$-norm of the system (4) such that $y(T) = 0$.

The proof of this theorem is similar to that in [9], also it can be done as the proof in subsection 4.2.

3 FULLY DISCRETE APPROXIMATIONS

Let $M, N \in \mathbb{N}$. We set $h = \frac{1}{N+1}$ and $\Delta t = \frac{T}{M+1}$ and introduce the nets

$$\begin{aligned} &0 = x_0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1, \\ &0 = t_0 < t_1 = \Delta t < \cdots < t_k = k\Delta t < \cdots < t_{M+1} = 1. \end{aligned} \tag{13}$$

We consider the following Crank-Nicolson discretization of (4)

$$\begin{cases} \frac{y_j^{n+1} - y_j^n}{\Delta t} + i\frac{y_{j+1}^{n+1} + y_{j-1}^{n+1} - 2y_j^{n+1}}{2h^2} + i\frac{y_{j+1}^n + y_{j-1}^n - 2y_j^n}{2h^2} = 0, & j = 1, \dots, N,\ n = 1, \dots, M, \\ y_0^n = 0, \quad \frac{y_{N+1}^{n+1} + y_{N+1}^n}{2} = v_h^n, & n = 1, \dots, M, \\ y_j^0 = y_{0j}, & j = 1, \dots, N. \end{cases} \tag{14}$$

We shall denote by $\tilde{y}^n = (y_1^n, \dots, y_N^n)$ the solution at the time step n. We consider also the system

$$\begin{cases} \frac{u_j^{n+1} - u_j^n}{\Delta t} + i\frac{u_{j+1}^{n+1} + u_{j-1}^{n+1} - 2u_j^{n+1}}{2h^2} + i\frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{2h^2} = 0, & j = 1, \dots, N,\ n = 1, \dots, M, \\ u_0^n = u_{N+1}^n = 0, & n = 1, \dots, M, \\ u_j^0 = u_{0j}, & j = 1, \dots, N. \end{cases} \tag{15}$$

Simple formal calculations give

$$\tilde{u}^{n+1} = (I - \frac{\Delta t}{2} iA_h)^{-1} (I + \frac{\Delta t}{2} iA_h) \tilde{u}^n = e^{i\alpha_{k,h}\Delta t} \tilde{u}^n,$$

where $\tilde{u}^n = (u_1^n, \dots, u_N^n)$ is the solution at the time step n and $e^{i\alpha_{k,h}\Delta t} = \frac{1 + \frac{\Delta t}{2} i\lambda_{k,h}}{1 - \frac{\Delta t}{2} i\lambda_{k,h}}$. Writing

$$\tilde{u}^0 = \sum_{k=1}^{N} a_k \tilde{\Phi}_k,$$

then the solution $\tilde{u}^n$ is given by

$$\tilde{u}^n = \sum_{k=1}^{N} a_k e^{i\alpha_{k,h} n\Delta t} \tilde{\Phi}_k, \tag{16}$$

with $a_k \in \mathbb{C}$, $\tilde{\Phi}_k = (\Phi_1^{k,h}, \dots, \Phi_N^{k,h}) = (\sin(k\pi h), \dots, \sin(Nk\pi h))$ and

$$\alpha_{k,h} = \frac{2}{\Delta t} \arctan\left(\frac{\lambda_{k,h}\Delta t}{2}\right).$$

The energy of (15) is

$$E^n = \frac{h}{2} \sum_{j=0}^{N} \left| \frac{u_{j+1}^n - u_j^n}{h} \right|^2,$$

which is a discretization of the continuous energy E in (2), and it is conserved in all the time steps: $E^n = E^0$, $n = 0, \dots, M$, for the solutions of (15).

3.1 Uniform observability of (15)

In this subsection, our goal is to prove the uniform observability inequality of system (15). We have the following theorem.

Theorem 6. *Let $0 < \gamma < 4$. Assume that*

$$\frac{h^2}{\Delta t} \leq \tau, \tag{17}$$

where τ is a positive constant. Then for any $0 < \delta < \frac{\gamma}{\tau}$, there exists T_δ such that for any $T > T_\delta$ there exists $C_{T,\delta,\gamma}$ such that the observability inequality

$$E^0 \leq C_{T,\delta,\gamma} \Delta t \sum_{n=0}^{M} \left| \frac{u_N^{n+1} + u_N^n}{2h} \right|^2 \tag{18}$$

holds for every solution of (15) with initial data in the class $C_{\delta/\Delta t}$ for all h and Δt small enough satisfying (17).

The proof of this Theorem will essentially rely on the following Theorem proved in [5].

Theorem 7. *Let $I = \mathbb{N}$ or $\mathbb{Z}$ and $(\mu_j)_{j\in\mathbb{N}}$ be an increasing sequence of real numbers such that, for some $\theta > 0$,*

$$\inf_{j\in I} |\mu_{j+1} - \mu_j| \geq \theta. \tag{19}$$

Let f be a smooth function satisfies the assumptions: $f \in C^\infty$ and satisfies $f(0) = 0, f'(0) = 1$; f is odd; $f : [-R, R] \to [-\pi, \pi]$, where $R \in \mathbb{R}_+^* \cup \{+\infty\}$; $\inf\{f'(\alpha) | |\alpha| \leq \delta\} > 0$, where $\delta \in (0, R)$. Then for all time

$$T > \frac{2\pi}{\theta \inf\limits_{|\alpha|\leq\delta} f'(\alpha)}$$

there exist two positive constants C and $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, for all $(a_j)_{j\in I} \in l^2(I)$ vanishing for $j \in I$ such that $|\mu_j|\tau \geq \delta$,

$$\frac{1}{C}\sum_{j\in I}|a_j|^2 \leq \tau \sum_{k\tau\in(0,T)} \left|\sum_{j\in I} a_j e^{if(\mu_j\tau)k}\right|^2 \leq C\sum_{j\in I}|a_j|^2.$$

Proof of Theorem 6. The energy of solutions (15) is

$$E^0 = \frac{h}{2}\sum_{j=0}^{N} \left| \frac{u_{j+1}^0 - u_j^0}{h} \right|^2 = \frac{h}{2} \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} |a_k|^2 \lambda_{k,h} \sum_{j=0}^{N} \left|\Phi_j^{k,h}\right|^2,$$

where we used

$$\sum_{j=0}^{N} \left| \frac{\Phi_{j+1}^{k,h} - \Phi_j^{k,h}}{h} \right|^2 = \lambda_{k,h} \sum_{j=0}^{N} \left|\Phi_j^{k,h}\right|^2.$$

Normalizing the eigenvector $\Phi^{k,h}$, i.e. $h\sum_{j=0}^{N}\left|\Phi_j^{k,h}\right|^2 = 1$, we get

$$E^0 = \frac{2}{h^2} \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} |a_k|^2 \sin^2\left(\frac{k\pi h}{2}\right) = \frac{2}{h^2} \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} |a_k|^2 \frac{\sin^2(k\pi h)}{4\cos^2(\frac{k\pi h}{2})} = \frac{1}{2} \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} |b_k|^2 \frac{4 + \lambda_{k,h}^2 \Delta t^2}{4\cos^2(\frac{k\pi h}{2})},$$

where

$$b_k = (-1)^k a_k (1 + e^{i\alpha_{k,h}\Delta t}) \frac{\sin(k\pi h)}{2h}.$$

Here we used the fact that

$$|1 + e^{i\alpha_{k,h}\Delta t}|^2 = 4\cos^2\left(\frac{\alpha_{k,h}\Delta t}{2}\right) = \frac{16}{4 + \lambda_{k,h}^2 \Delta t^2}.$$

In virtue of (17), we have $C_{\frac{\delta}{\Delta t}} \subset C_{\frac{\gamma}{h^2}}$ and then we get

$$\frac{1}{4\cos^2(\frac{\alpha_{k,h}\Delta t}{2})} \leq \frac{1}{4-\gamma} \text{ and } 4 + \lambda_{k,h}^2 \Delta t^2 \leq 4 + \delta^2.$$

On the other hand, we have

$$\begin{aligned} \Delta t \sum_{n=0}^{M} \left| \frac{u_N^{n+1} + u_N^n}{2h} \right|^2 &= \Delta t \sum_{n=0}^{M} \left| \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} a_k e^{i\alpha_{k,h} n\Delta t} (1 + e^{i\alpha_{k,h}\Delta t}) \frac{\varphi_N^{|k|,h}}{2h} \right| \\ &= \Delta t \sum_{n=0}^{M} \left| \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} b_k e^{i\alpha_{k,h} n\Delta t} \right|^2 = \Delta t \sum_{n=0}^{M} \left| \sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} b_k e^{if(\lambda_{k,h}\Delta t)n} \right|^2, \end{aligned}$$

where $f(t) = 2\arctan(\frac{t}{2})$. It is clear that the function f satisfies the assumptions of Theorem 7. Besides, it was proved in [9] that for all $\varepsilon \in (0, 1)$, we have

$$\lambda_{k+1,h} - \lambda_{k,h} \geq 3\pi^2 - \varepsilon.$$

Consequently (19) is verified with $\theta = 3\pi^2 - \varepsilon$. Applying Theorem 7, we obtain

$$E^0 \leq \frac{C(4+\delta^2)}{4-\gamma} \Delta t \sum_{n=0}^{M} \left| \frac{u_N^{n+1} + u_N^n}{2h} \right|^2,$$

for all $T > T_\delta = \frac{\pi(4+\delta^2)}{2\theta}$. □

3.2 Uniform controllability of (14)

In this part, we present the following uniform partial controllability result for system (14) and the convergence result for the controls.

The partial controllability problem for system (14) in the space H_h^{-1} consists of finding a control $(v_h^n)_{0,1,\dots,M}$ such that for all initial data $\tilde{y}^0 \in H_h^{-1}$ the solution $\tilde{y}^n$ of (14) satisfies

$$P_\delta \tilde{y}^{M+1} = 0,$$

where δ is the same in Theorem 6 and P_δ is the orthogonal projection over $C_{\delta/\Delta t}$.

The main result of this paper reads as follows.

Theorem 8. *Let T, γ, τ and δ be given as in Theorem 6. Then for every Δt and h small enough and every $y^0 \in H^{-1}(0, 1)$, the system (14) is partially controllable on H_h^{-1} with controls v_h^n. Moreover, we have:*

i) the controls of minimal norm are uniformly bounded with respect to Δt;

ii) the controls v_h^n converge to a control v of the minimal L^2-norm of the controllable system (4).

Proof. For any given $T > T_\delta$, choose γ, τ and δ as in Theorem 6 to guarantee the uniform observability (18). Multiplying the first equation in (14) by a solution $\frac{\bar{u}_j^{n+1}+\bar{u}_j^n}{2}$ of (15), adding in j and n and taking the imaginary parts, we get

$$\text{Re}\ \Delta t \sum_{n=0}^{M} v_h^n \frac{\bar{u}_N^{n+1} + \bar{u}_N^n}{2h} - \text{Im}\ h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 = 0. \tag{20}$$

Let $\tilde{u}^n \in C_{\delta/\Delta t}$ be the solution of (15) with initial data $\tilde{u}^0$ and define the functional $J_{h,\Delta t} : \mathbb{R}^n \to \mathbb{R}$ by

$$J_{h,\Delta t}(\tilde{u}^0) = \frac{\Delta t}{2} \sum_{n=0}^{M} \left| \frac{u_N^{n+1} + u_N^n}{2h} \right|^2 - \text{Im}\ h \sum_{j=0}^{N} y_j^0 \tilde{u}_j^0.$$

For $\tilde{u}^n \in C_{\delta/\Delta t}$ we have

$$\left| \text{Im}\ h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 \right| \le \left| (P_\delta \tilde{y}^0, \bar{\tilde{u}}^0)_{\mathbb{R}^N} \right| \le \|P_\delta \tilde{y}^0\|_{H_h^{-1}} \|\tilde{u}^0\|_{H_h^1}. \tag{21}$$

The functional $J_{h,\Delta t}$ is continuous and convex. Moreover, in view of the observability inequality (18), it is clear that $J_{h,\Delta t}$ is coercive. Thus, there exists unique minimizer $\hat{u}^0$ of $J_{h,\Delta t}$,

$$J_{h,\Delta t}(\hat{u}^0) = \min_{\tilde{u}^0 \in C_{\delta/\Delta t}} J_{h,\Delta t}(\tilde{u}^0).$$

Let $\hat{u}^n \in C_{\delta/\Delta t}$ be the solution of the system (15) with initial data $\hat{u}^0$. The $\hat{u}^0$ satisfies the Euler-Lagrange equation. Calculating The Gateaux derivative of $J_{h,\Delta t}$ in $\hat{u}^0$, we get

$$0 = \lim_{t \to 0} \frac{J_{h,\Delta t}(\hat{u}^0 + t\bar{u}^0) - J_{h,\Delta t}(\hat{u}^0)}{t} = \text{Re}\ \Delta t \sum_{n=0}^{M} \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \frac{\bar{u}_N^{n+1} + \bar{u}_N^n}{2h} - \text{Im}\ h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0.$$

Therefore, according to (20) we choose the control function v_h^n in system (14) as follows

$$v_h^n = \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h}, \quad n = 0, \ldots, M.$$

We now check the uniform boundedness of the controls v_h^n. We have

$$J_{h,\Delta t}(\hat{u}^0) \le J_{h,\Delta t}(0) = 0,$$

and by (21), we get

$$\frac{\Delta t}{2} \sum_{n=0}^{M} \left| \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \right|^2 \le \|P_\delta \tilde{y}^0\|_{H_h^{-1}} \|\hat{u}^0\|_{H_h^1}.$$

Applying the observability inequality (18) we obtain

$$\Delta t \sum_{n=0}^{M} \left| \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \right|^2 \le 2\sqrt{\frac{2C(4+\delta^2)}{4-\gamma}} \|P_\delta \tilde{y}^0\|_{H_h^{-1}} \left(\Delta t \sum_{n=0}^{M} \left| \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \right|^2 \right)^{\frac{1}{2}},$$

where we used

$$E^0 = \frac{1}{2} \|\hat{u}^0\|_{H_h^1}.$$

Consequently, the controls $v_h^n = \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h}$ satisfy

$$\left(\Delta t \sum_{n=0}^{M} |v_h^n|^2 \right)^{\frac{1}{2}} \le C(T, \delta, \gamma) \|P_\delta \tilde{y}^0\|_{H_h^{-1}}.$$

Therefore, the controls are uniformly bounded with respect to Δt.

Let us now give some details for the proof of the convergence result. Indeed the proof is standard and one may use the method developed in [12]. Note that with the notations (16), the controls (v_h^n) are of the form

$$\frac{1}{2h} \sum_{\lambda_{k,h} \le \delta/\Delta t} m_k e^{i\alpha_{k,h} n \Delta t} (1 + e^{i\alpha_{k,h}\Delta t}) \sin(k\pi N h),$$

where $(m_k)_k$ are the Fourier coefficients of the solution $\hat{u}^n \in C_{\delta/\Delta t}$ of (15), with initial data $\hat{u}^0$ being the minimizer of the functional $J_{h,\Delta t}$.

We define the continuous extension of the discrete controls by

$$v_h(t) = \frac{1}{2h} \sum_{\lambda_{k,h} \le \delta/\Delta t} m_k e^{i\alpha_{k,h} t} (1 + e^{i\alpha_{k,h}\Delta t}) \sin(k\pi N h).$$

Now, from the boundedness of (v_h^n), we see that, extracting subsequences, for some $v \in L^2(0,T)$ and $\hat{u}^0 \in H_0^1(0,1)$, $v_h \to v$ weakly in $L^2(0,T)$, $\hat{u}_h^0 \to \hat{u}^0$ weakly in $H_0^1(0,1)$, as $\Delta t \to 0$.

Moreover, one can show by standard arguments, that

$$v = -\hat{u}_x(1,t),$$

where $\hat{u}$ is the solution of (1) with initial data $\hat{u}^0 \in H_0^1(0,1)$, the unique minimizer of the functional J given in (5). Letting $\Delta t \to 0$ and $\Delta x \to 0$ in (20), we get

$$\text{Re} \int_0^T v\bar{u}_x(1)dt + \text{Im} \int_0^1 y^0 \bar{u}^0 dx = 0,$$

and this later condition implies that the solution of system (4) with control v given as above satisfies $y(T) = 0$.

On the other hand, taking into account the convergence of the linear term of the discrete functional $J_{h,\Delta t}$ to the linear term of the discrete continuous functional J, and the structure of J and $J_{h,\Delta t}$, we deduce that

$$\int_0^T |v_h|^2 dt \to \int_0^T |v|^2 dt \quad \text{as } \Delta \to 0.$$

This combined with the weak convergence ensure the strong convergence desired. □

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У статті досліджується керованість системи скінченної розмірності, яка отримана в результаті дискретизації в просторі та часі лінійного 1-D рівняння Шредінгера з граничним контролем. Як і для інших задач, можна очікувати, що рівномірна керованість не виконується у загальному випадку у зв'язку з високою частотою появи некоректних моделей. Базуючись на рівномірній граничній спостережуваній оцінці для фільтрованих розв'язків відповідної консервативної дискретної системи, показано рівномірну керованість проекції розв'язків на простір, породжений рештою власних форм.

Ключові слова і фрази: спостережуваність, контрольованість, фільтрування.

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PALEY–WIENER-TYPE THEOREM FOR POLYNOMIAL ULTRADIFFERENTIABLE FUNCTIONS

The image of the space of ultradifferentiable functions with compact supports under Fourier-Laplace transformation is described. An analogue of Paley-Wiener theorem for polynomial ultradifferentiable functions is proved.

Key words and phrases: ultradifferentiable function, ultradistribution, polynomial test function, Paley–Wiener-type theorem.

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INTRODUCTION

In general Paley–Wiener theorem is any theorem that relates decay properties of a function or distribution at infinity with analyticity of its Fourier transform [16]. For example, the Paley–Wiener theorem for the space of smooth functions with compact supports gives a characterization of its image as rapidly decreasing functions having a holomorphic extension to $\mathbb{C}$ of exponential type.

There are plenty of Paley–Wiener-type theorems since there are many kinds of bound for decay rates of functions and many types of characterizations of smoothness. In this regard a wide number of papers have been devoted to the extension of the theory on many other integral transforms and different classes of functions (see [1–3, 6, 9, 15, 17, 18, 20–22] and the references given there).

Let $\mathcal{G}'_\beta := \mathcal{G}'_\beta(\mathbb{R}^d)$ be the space of Roumieu ultradistributions on $\mathbb{R}^d$ and $\mathcal{G}_\beta := \mathcal{G}_\beta(\mathbb{R}^d)$ be its predual. A Fréchet-Schwartz space (briefly, (FS) space) is one that is Fréchet and Schwartz simultaneously (see [23]). It is known (see e.g. [10, 19]) that the spaces $\mathcal{G}'_\beta$ and $\mathcal{G}_\beta$ are nuclear Fréchet-Schwartz and dual Fréchet-Schwartz spaces ((DFS) for short), respectively. These facts are crucial for our investigation.

In this article we consider Fourier-Laplace transformation, defined on the space $\mathcal{G}_\beta$ of ultradifferentiable functions and on the corresponding algebra $\mathcal{P}(\mathcal{G}'_\beta)$ of polynomials over $\mathcal{G}'_\beta$ [12], which have the tensor structure of the form $\bigoplus_{fin} \mathcal{G}_\beta^{\hat{\otimes} n}$ (see Theorem 1).

We completely describe the image of test space $\mathcal{G}_\beta$ under Fourier-Laplace transformation (see Corollary 1 and Theorem 2) and prove Paley–Wiener-type Theorem 3 for polynomial ultradifferentiable functions.

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1 PRELIMINARIES AND NOTATIONS

Let $\mathscr{L}(X)$ denote the space of continuous linear operators over a locally convex space X and let X' be the dual of X. Throughout, we will endow $\mathscr{L}(X)$ and X' with the locally convex topology of uniform convergence on bounded subsets of X.

Let $\otimes_{\mathfrak{p}}$ denote completion of algebraic tensor product with respect to the projective topology $\mathfrak{p}$. Let $X^{\hat{\otimes}n}$, $n \in \mathbb{N}$, be the symmetric nth tensor degree of X, completed in the projective tensor topology. Note, that here and subsequently we omit the index $\mathfrak{p}$ to simplify notations. For any $x \in X$ we denote $x^{\otimes n} := \underbrace{x \otimes \cdots \otimes x}_{n} \in X^{\hat{\otimes}n}$, $n \in \mathbb{N}$. Set $X^{\hat{\otimes}0} := \mathbb{C}$, $x^{\otimes 0} := 1 \in \mathbb{C}$.

To define the locally convex space $\mathcal{P}_n(\mathcal{X}')$ of n-homogeneous polynomials on $\mathcal{X}'$ we use the canonical topological linear isomorphism

$$\mathcal{P}_n(\mathcal{X}') \simeq (\mathcal{X}'^{\hat{\otimes}n})'$$

described in [4]. Namely, given a functional $p_n \in (\mathcal{X}'^{\hat{\otimes}n})'$, we define an n-homogeneous polynomial $P_n \in \mathcal{P}_n(\mathcal{X}')$ by $P_n(x) := p_n(x^{\otimes n})$, $x \in \mathcal{X}'$. We equip $\mathcal{P}_n(\mathcal{X}')$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{X}'$. Set $\mathcal{P}_0(\mathcal{X}') := \mathbb{C}$. The space $\mathcal{P}(\mathcal{X}')$ of all continuous polynomials on $\mathcal{X}'$ is defined to be the complex linear span of all $\mathcal{P}_n(\mathcal{X}')$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Denote

$$\Gamma(\mathcal{X}) := \bigoplus_{n\in\mathbb{Z}_+}{}_{fin} \mathcal{X}^{\hat{\otimes}n} \subset \bigoplus_{n\in\mathbb{Z}_+} \mathcal{X}^{\hat{\otimes}n}.$$

Note, that we consider only the case when the elements of direct sum consist of finite but not fixed number of addends. For simplicity of notation we write $\Gamma(\mathcal{X})$ instead of commonly used $\Gamma_{fin}(\mathcal{X})$.

We have the following assertion (see also [12, Proposition 2.1]).

Theorem 1. *There exists the linear topological isomorphism*

$$\mathrm{Y}_{\mathcal{X}} \colon \Gamma(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X}')$$

for any nuclear (F) or (DF) space $\mathcal{X}$.

Let $A : X \longrightarrow Y$ be any linear and continuous operator, where X, Y are locally convex spaces. It is easy to see, that the operator $A \otimes I_Y$, defined on the tensor product $X \otimes Y$ by the formula

$$(A \otimes I_Y)(x \otimes y) := Ax \otimes y, \qquad x \in X, \quad y \in Y,$$

is linear, where I_Y denotes the identity on Y. The operator $A \otimes I_Y$ is continuous in projective topology $\mathfrak{p}$ and it has a unique extension to linear continuous operator onto the space $X \otimes_{\mathfrak{p}} Y$.

The following assertion essentially will be used in the proof of Theorem 3.

Proposition 1 ([13]). *For any nuclear (F) or (DF) spaces X, Y, and any operator $A \in \mathscr{L}(X,Y)$ the following equality holds*

$$\ker(A \otimes I_Y) = \ker(A) \otimes_{\mathfrak{p}} Y.$$

2 SPACES OF FUNCTIONS

Let us consider the definition and some properties of the space of Gevrey ultradifferentiable functions with compact supports. For more details we refer the reader to [10, 11].

We use the following notations: $t^k := t_1^{k_1} \cdot \ldots \cdot t_d^{k_d}$, $k^{k\beta} := k_1^{k_1\beta} \cdot \ldots \cdot k_d^{k_d\beta}$, $|k| := k_1 + \cdots + k_d$ for all $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$(or $\mathbb{C}^d$), $k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ and $\beta > 1$. Let $\partial^k := \partial_1^{k_1} \ldots \partial_d^{k_d}$, where $\partial_j^{k_j} := \partial^{k_j}/\partial t_j^{k_j}$, $j = 1, \ldots, d$. The notation $\mu \prec \nu$ with $\mu, \nu \in \mathbb{R}^d$ means that $\mu_1 < \nu_1, \ldots, \mu_d < \nu_d$ (similarly, $\mu \succ \nu$). Let $[\mu, \nu] := [\mu_1, \nu_1] \times \ldots \times [\mu_d, \nu_d]$ and $(\mu, \nu) := (\mu_1, \nu_1) \times \ldots \times (\mu_d, \nu_d)$ for any $\mu \prec \nu$. In the following $t \in [\mu, \nu]$ means that $t_j \in [\mu_j, \nu_j]$ and $t \to \infty$ (resp. $t \to 0$) means that $t_j \to \infty$ (resp. $t_j \to 0$) for all $j = 1, \ldots, d$.

A complex infinitely smooth function φ on $\mathbb{R}^d$ is called a Gevrey ultradifferentiable with $\beta > 1$ (see [10, II.2.1]) if for every $[\mu, \nu] \subset \mathbb{R}^d$ there exist constants $h > 0$ and $C > 0$ such that

$$\sup_{t\in[\mu,\nu]} |\partial^k \varphi(t)| \le C h^{|k|} k^{k\beta} \tag{1}$$

holds for all $k \in \mathbb{Z}_+^d$.

For a fixed $h > 0$, consider the subspace $\mathcal{G}_\beta^h[\mu, \nu]$ of all functions supported by $[\mu, \nu] \subset \mathbb{R}^d$ and such that there exists a constant $C = C(\varphi) > 0$, that inequality (1) holds for all $k \in \mathbb{Z}_+^d$. Therefore, the space of ultradifferentiable functions with compact supports is defined as follows

$$\mathcal{G}_\beta^h[\mu, \nu] := \{\varphi \in C^\infty(\mathbb{R}^d) : \operatorname{supp} \varphi \subset [\mu, \nu], \|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]} < \infty\},$$

with the norm

$$\|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+^d} \sup_{t\in[\mu,\nu]} \frac{|\partial^k \varphi(t)|}{h^{|k|} k^{k\beta}}.$$

Proposition 2 ([10]). *Each $\mathcal{G}_\beta^h[\mu, \nu]$ is a Banach space, and all inclusions $\mathcal{G}_\beta^h[\mu, \nu] \hookrightarrow \mathcal{G}_\beta^l[\mu, \nu]$ with $h < l$ are compact. Moreover, if $[\mu, \nu] \subset [\mu', \nu']$, then $\mathcal{G}_\beta^h[\mu, \nu]$ is closed subspace in $\mathcal{G}_\beta^h[\mu', \nu']$.*

This proposition implies that the set of Banach spaces

$$\left\{\mathcal{G}_\beta^h[\mu, \nu] : [\mu, \nu] \subset \mathbb{R}^d, \ h > 0\right\}$$

is partially ordered. Therefore we can consider this set as inductive system with respect to stated above compact inclusions. Hence, we define the space

$$\mathcal{G}_\beta(\mathbb{R}^d) := \bigcup_{\mu\prec\nu,\, h>0} \mathcal{G}_\beta^h[\mu, \nu], \qquad \mathcal{G}_\beta(\mathbb{R}^d) \simeq \liminf_{\mu\prec\nu,\, h>0} \mathcal{G}_\beta^h[\mu, \nu],$$

and endow it with the topology of inductive limit.

The strong dual space $\mathcal{G}_\beta'(\mathbb{R}^d)$ is called the space of Roumieu ultradistributions on $\mathbb{R}^d$.

Proposition 3 ([10]). *The spaces $\mathcal{G}_\beta(\mathbb{R}^d)$ and $\mathcal{G}_\beta'(\mathbb{R}^d)$ are nonempty locally convex nuclear reflexive spaces. Moreover, $\mathcal{G}_\beta(\mathbb{R}^d)$ is (DFS) space, and $\mathcal{G}_\beta'(\mathbb{R}^d)$ is (FS) space.*

Next define the space of entire functions of exponential type, which will be an image of the space $\mathcal{G}_\beta(\mathbb{R}^d)$ under the Fourier-Laplace transformation (see Section 3).

Let M be a set in $\mathbb{R}^d$. The support function of the set M is defined to be a function

$$H_M(x) = \sup_{t\in M}(t,x), \qquad x \in \mathbb{R}^d,$$

where $(t,x) := t_1x_1 + \dots + t_dx_d$ denotes the scalar product. It is known [7], that $H_M(\eta)$ is convex, lower semi-continuous function, that may take the value $+\infty$. If M is bounded set, then its support function is continuous.

Let $B_r \subset \mathbb{C}^d$ be a ball of a radius $r > 0$. The space $E(\mathbb{C}^d)$ of entire functions of exponential type we will endow with locally convex topology of uniform convergence on compact sets. This topology can be defined by the system of seminorms

$$p_{r,M}(\psi) := \sup_{z\in B_r} |\psi(z)|e^{-H_M(\eta)},$$

where $\eta = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d$ is imaginary part of $z = (z_1, \dots, z_d) \in \mathbb{C}^d$.

Fix an arbitrary real $\beta > 1$. For a positive number $h > 0$ and vectors $\mu = (\mu_1, \dots, \mu_d)$, $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{R}^d$, such that $\mu \prec \nu$, in the space of entire functions of exponential type we define the subspace $E^h_\beta[\mu,\nu]$ of functions $\mathbb{C}^d \ni z \longmapsto \psi(z) \in \mathbb{C}$ with finite norm

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}^d_+} \sup_{z\in\mathbb{C}^d} \frac{|z^k\psi(z)e^{-H_{[\mu,\nu]}(\eta)}|}{h^{|k|}k^{k\beta}}. \tag{2}$$

Since for any $r > 0$ and $\psi \in E^h_\beta[\mu,\nu]$ the next inequality $p_{r,[\mu,\nu]}(\psi) \le \|\psi\|_{E^h_\beta[\mu,\nu]}$ is valid, then all inclusions $E^h_\beta[\mu,\nu] \hookrightarrow E(\mathbb{C}^d)$ are continuous.

Proposition 4. *Each space $E^h_\beta[\mu,\nu]$ is Banach space, and all inclusions*

$$E^h_\beta[\mu,\nu] \hookrightarrow E^{h'}_\beta[\mu',\nu'] \qquad \text{with} \qquad [\mu,\nu] \subset [\mu',\nu'],\ h < h',$$

are compact.

Proof. Let us prove the completeness of the space $E^h_\beta[\mu,\nu]$. Let $\{\psi_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence in $E^h_\beta[\mu,\nu]$. It means that for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \in \mathbb{N}$ such that $\forall\, m,n > N_\varepsilon$ the next inequality $\|\psi_m - \psi_n\|_{E^h_\beta[\mu,\nu]} < \varepsilon$ is valid.

The following inequality

$$\sup_{z\in B_r} \frac{|z^k\psi(z)|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} \le \|\psi\|_{E^h_\beta[\mu,\nu]}, \qquad \psi \in E^h_\beta[\mu,\nu],$$

is obvious for all $k \in \mathbb{Z}^d_+$ and $r > 0$. It follows that $\{\varphi_m\}_{m\in\mathbb{N}}$, where $\varphi_m(z) := \dfrac{z^k\psi_m(z)}{h^{|k|}k^{k\beta}}$, is fundamental sequence in the space of entire functions of exponential type. Therefore for any $k \in \mathbb{Z}^d_+$ and $r > 0$ we have

$$\sup_{z\in B_r} \frac{|z^k(\psi_m(z) - \psi_n(z))|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} < \varepsilon, \quad \forall\, m,n > N_\varepsilon. \tag{3}$$

Since $\{\varphi_m\}_{m\in\mathbb{N}}$ is fundamental sequence, it is bounded in $E(\mathbb{C}^d)$. From the Bernstein theorem on compactness [14, theorem 3.3.6] it follows that there exist a subsequence $\{\varphi_{k_m}\}_{k_m\in\mathbb{N}}$ and a function $\varphi \in E(\mathbb{C}^d)$ such that the following equality is satisfied

$$\lim_{k_m\to\infty} \sup_{z\in B_r} \frac{|z^k(\psi_{k_m}(z) - \psi(z))|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} = 0, \qquad k \in \mathbb{Z}^d_+, \quad r > 0.$$

Let us pass to the limit in (3) as $m = k_m \to \infty$. Consequently, for all $k \in \mathbb{Z}^d_+$ and $r > 0$ we obtain the inequality

$$\sup_{z\in B_r} \frac{|z^k(\psi(z) - \psi_n(z))|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} < \varepsilon,$$

which satisfies for all $n > N_\varepsilon$. Hence from the triangle inequality we obtain

$$\sup_{z\in B_r} \frac{|z^k\psi(z)|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} \le \sup_{z\in B_r} \frac{|z^k\psi_{n_0}(z)|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} + \varepsilon,$$

where $n_0 = N_\varepsilon + 1$.

Taking a supremum over k and r in the above inequality, we obtain

$$\|\psi\|_{E^h_\beta[\mu,\nu]} \le \|\psi_{n_0}\|_{E^h_\beta[\mu,\nu]} + \varepsilon,$$

therefore $\psi \in E^h_\beta[\mu,\nu]$. Hence, the space $E^h_\beta[\mu,\nu]$ is complete.

The compactness of inclusions $E^h_\beta[\mu,\nu] \hookrightarrow E^{h'}_\beta[\mu',\nu']$ with $[\mu,\nu] \subset [\mu',\nu']$, $h < h'$ follows from obvious inequality $e^{-H_{[\mu',\nu']}} \le e^{-H_{[\mu,\nu]}}$ and from [10, pp. 38–40]. □

Define the space

$$E_\beta(\mathbb{C}^d) := \bigcup_{\mu\prec\nu,\, h>0} E^h_\beta[\mu,\nu], \qquad E_\beta(\mathbb{C}^d) \simeq \liminf_{\mu\prec\nu,\, h>0} E^h_\beta[\mu,\nu],$$

and endow it with the topology of inductive limit with respect to compact inclusions from the Proposition 4.

In what follows to simplify the notations we will write $\mathcal{G}_\beta := \mathcal{G}_\beta(\mathbb{R}^d)$, $\mathcal{G}'_\beta := \mathcal{G}'_\beta(\mathbb{R}^d)$, $E_\beta := E_\beta(\mathbb{C}^d)$, $E'_\beta := E'_\beta(\mathbb{C}^d)$.

3 FOURIER-LAPLACE TRANSFORM AND PALEY–WIENER-TYPE THEOREM

Consider the inductive limits of Banach spaces

$$E_\beta[\mu,\nu] := \bigcup_{h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad E_\beta[\mu,\nu] \simeq \liminf_{h\to\infty} \mathcal{G}^h_\beta[\mu,\nu],$$

and

$$\mathcal{G}_\beta[\mu,\nu] := \bigcup_{h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad \mathcal{G}_\beta[\mu,\nu] \simeq \liminf_{h\to\infty} \mathcal{G}^h_\beta[\mu,\nu].$$

On the space $\mathcal{G}_\beta$ we define the Fourier-Laplace transform

$$\widehat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}^d} e^{-i(t,z)}\varphi(t)\,dt, \qquad \varphi \in \mathcal{G}_\beta, \quad z \in \mathbb{C}^d. \tag{4}$$

Our main task is to show, that the function $\widehat{\varphi}(z)$ belongs to the space E_β, moreover, we will prove that the mapping $F : \mathcal{G}_\beta \longrightarrow E_\beta$ is surjective. For this end we prove the following auxiliary statement, which can be found in [8, Lemma 1], but our proof is different.

Proposition 5. *The image of the space $\mathcal{G}_\beta[\mu,\nu]$ with respect to mapping F is the space $E_\beta[\mu,\nu]$.*

Proof. Let $\varphi \in \mathcal{G}_\beta[\mu,\nu]$. Properties of the Fourier transform imply $\widehat{\partial^k\varphi}(z) = z^k\widehat{\varphi}(z)$ for all $k \in \mathbb{Z}^d_+$. Therefore for any z and k we have

$$|z^k\widehat{\varphi}(z)| = \Big|\int_{\mathbb{R}^d} e^{-i(t,z)}\partial^k\varphi(t)\,dt\Big| \le \int_{[\mu,\nu]} |e^{-i(t,\xi)}e^{(t,\eta)}\partial^k\varphi(t)|\,dt$$
$$\le h^{|k|}k^{k\beta}e^{H_{[\mu,\nu]}(\eta)}\|\varphi\|_{\mathcal{G}^h_\beta[\mu,\nu]}\int_{[\mu,\nu]} dt.$$

It follows

$$\|\widehat{\varphi}\|_{E^h_\beta[\mu,\nu]} \le C\|\varphi\|_{\mathcal{G}^h_\beta[\mu,\nu]}, \tag{5}$$

where $C = \prod_{j=1}^d(\nu_j - \mu_j)$. Hence, $F(\mathcal{G}^h_\beta[\mu,\nu]) \subset E^h_\beta[\mu,\nu]$.

Vice versa. Let $\psi \in E^h_\beta[\mu,\nu]$. It is known, that the norm of the space $E^h_\beta[\mu,\nu]$ can be defined by the formula

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}^d_+}\sup_{z\in\mathbb{C}^d}\frac{|z^k\psi(z)e^{-H_{[\mu,\nu]}(\eta)}|}{h^{|k|}|k|!^\beta},$$

moreover, the topology, defined by this norm, is equivalent to earlier defined (see (2)). It follows that for each function $\psi \in E^h_\beta[\mu,\nu]$ there exists a constant C such that the inequality

$$|z^k\psi(z)| \le Ch^{|k|}|k|!^\beta e^{H_{[\mu,\nu]}(\eta)} \tag{6}$$

holds for all $z \in \mathbb{C}^d$.

The following inequality

$$e^{\beta t^{1/\beta}} = (e^{t^{1/\beta}})^\beta = \Big(\sum_{m=0}^{\infty}\frac{t^{m/\beta}}{m!}\Big)^\beta \ge \frac{|t|^m}{m!^\beta},$$

holds for all $t \in \mathbb{R}$ and $m \in \mathbb{Z}_+$. In particular, for $t = |z|/h$ and $m = |k|$, we obtain

$$e^{\beta\left(\frac{|z|}{h}\right)^{1/\beta}} \ge \frac{|z|^{|k|}}{h^{|k|}|k|!^\beta}.$$

Hence from the inequality $|z^k| \le |z|^{|k|}$ it follows

$$\frac{h^{|k|}|k|!^\beta}{|z^k|}e^{H_{[\mu,\nu]}(\eta)} \ge \frac{e^{H_{[\mu,\nu]}(\eta)}}{e^{(L|z|)^{1/\beta}}},$$

where $L = \frac{\beta^\beta}{h}$. So, if the function ψ satisfies the inequality (6), i.e. belongs to the space $E^h_\beta[\mu,\nu]$, then it satisfies the inequality

$$|\psi(z)| \le Ce^{-(L|z|)^{1/\beta}+H_{[\mu,\nu]}(\eta)}.$$

From the theorem [10, theorem 2.22] it follows that there exists a function $\varphi \in \mathcal{G}_\beta[\mu,\nu]$ such that $\widehat{\varphi} = \psi$, i.e. $E^h_\beta[\mu,\nu] \subset F(\mathcal{G}^h_\beta[\mu,\nu])$.

Hence, we have proved $F(\mathcal{G}^h_\beta[\mu,\nu]) = E^h_\beta[\mu,\nu]$. Since the constant $h > 0$ is arbitrary, properties of inductive limit imply the desired equality

$$F(\mathcal{G}_\beta[\mu,\nu]) = E_\beta[\mu,\nu].$$

□

The immediate consequence of the Proposition 5 and of the properties of inductive limit is the following assertion.

Corollary 1. *The image of the space $\mathcal{G}_\beta$ with respect to mapping F is the space E_β.*

Therefore, we may consider the adjoint mapping $F' : E'_\beta \longrightarrow \mathcal{G}'_\beta$.

Theorem 2. *There exist the following topological isomorphisms*

$$F(\mathcal{G}_\beta) \simeq E_\beta \qquad \textit{and} \qquad F'(E'_\beta) \simeq \mathcal{G}'_\beta.$$

Proof. The inequality (5) implies, that the mapping

$$F : \mathcal{G}_\beta[\mu,\nu] \ni \varphi \longmapsto \widehat{\varphi} \in E_\beta[\mu,\nu]$$

is continuous. From the Proposition 5 we obtain the surjectivity of the map. Therefore, the open map theorem [5, theorem 6.7.2] implies the topological isomorphism $F(\mathcal{G}_\beta[\mu,\nu]) \simeq E_\beta[\mu,\nu]$. Since the segment $[\mu,\nu]$ is arbitrary, the properties of inductive limit imply the desired topological isomorphisms. □

Using the Theorem 1 and a tensor structure of the space

$$\Gamma(\mathcal{G}_\beta) := \bigoplus\nolimits_{fin\, n\in\mathbb{Z}_+}\mathcal{G}_\beta^{\widehat{\otimes}n} \subset \bigoplus_{n\in\mathbb{Z}_+}\mathcal{G}_\beta^{\widehat{\otimes}n},$$

we extend the mapping F to the mapping $F^\otimes$, that defined on $\Gamma(\mathcal{G}_\beta)$.

At first, take an element $\varphi^{\otimes n} \in \mathcal{G}_\beta^{\widehat{\otimes}n}$, with $\varphi \in \mathcal{G}_\beta$, from the total subset of $\mathcal{G}_\beta^{\widehat{\otimes}n}$. Define the operator $F^{\otimes n}$ as follows

$$F^{\otimes n} : \varphi^{\otimes n} \longmapsto \widehat{\varphi}^{\otimes n} \quad \text{and} \quad F^{\otimes 0} := I_{\mathbb{C}},$$

where $\widehat{\varphi}^{\otimes n} := (F\varphi)^{\otimes n}$. Next, we extend the map $F^{\otimes n}$ onto whole space $\mathcal{G}_\beta^{\widehat{\otimes}n}$ by linearity and continuity. So, we obtain $F^{\otimes n} \in \mathscr{L}(\mathcal{G}_\beta^{\widehat{\otimes}n}, E_\beta^{\widehat{\otimes}n})$. Finally, we define $F^\otimes$ as the mapping

$$F^\otimes := (F^{\otimes n}) : \Gamma(\mathcal{G}_\beta) \ni \boldsymbol{p} := (p_n) \quad\longmapsto\quad \widehat{\boldsymbol{p}} := (\widehat{p}_n) \in \Gamma(E_\beta), \tag{7}$$

where $p_n \in \mathcal{G}_\beta^{\widehat{\otimes}n}$, $\widehat{p}_n := F^{\otimes n}p_n \in E_\beta^{\widehat{\otimes}n}$.

The following commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{G}'_\beta) & \xrightarrow{F^\otimes_{\mathcal{P}}} & \mathcal{P}(E'_\beta) \\ \Upsilon^{-1}_{\mathcal{G}_\beta}\downarrow & & \uparrow\Upsilon_{E_\beta} \\ \Gamma(\mathcal{G}_\beta) & \xrightarrow{F^\otimes} & \Gamma(E_\beta) \end{array} \tag{8}$$

uniquely defines the operator $F^\otimes_{\mathcal{P}} : \mathcal{P}(\mathcal{G}'_\beta) \longrightarrow \mathcal{P}(E'_\beta)$. The map $F^\otimes_{\mathcal{P}}$ we will call the polynomial Fourier-Laplace transformation.

We proved above that the mappings $F : \mathcal{G}_\beta \longrightarrow E_\beta$ and $F' : E'_\beta \longrightarrow \mathcal{G}'_\beta$ are topological isomorphisms. Let us prove the analogue of this result. The next theorem may be considered as Paley–Wiener-type theorem.

Theorem 3. *Polynomial Fourier-Laplace transformation is topological isomorphism from the algebra $\mathcal{P}(\mathcal{G}'_\beta)$ into the algebra $\mathcal{P}(E'_\beta)$.*

Proof. From the Theorem 1 and commutativity of the diagram (8) it follows that it is enough to show that the mapping $F^{\otimes} : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta)$ is topological isomorphism.

Theorem 2 and Corollary 1 imply the following equalities

$$\ker F = \{0\}, \qquad \ker F^{-1} = \{0\}.$$

Let us consider the operators

$$I_{\mathcal{G}_\beta} \otimes F : \mathcal{G}_\beta \otimes \mathcal{G}_\beta \longrightarrow \mathcal{G}_\beta \otimes E_\beta, \qquad F \otimes I_{E_\beta} : \mathcal{G}_\beta \otimes E_\beta \longrightarrow E_\beta \otimes E_\beta,$$
$$I_{E_\beta} \otimes F^{-1} : E_\beta \otimes E_\beta \longrightarrow E_\beta \otimes \mathcal{G}_\beta, \qquad F^{-1} \otimes I_{\mathcal{G}_\beta} : E_\beta \otimes \mathcal{G}_\beta \longrightarrow \mathcal{G}_\beta \otimes \mathcal{G}_\beta.$$

Since spaces $\mathcal{G}_\beta$ and E_β are nuclear (DF) spaces, Proposition 1 implies the equalities

$$\ker(I_{\mathcal{G}_\beta} \otimes F) = \{0\}, \qquad \ker(F \otimes I_{E_\beta}) = \{0\},$$
$$\ker(I_{E_\beta} \otimes F^{-1}) = \{0\}, \qquad \ker(F^{-1} \otimes I_{\mathcal{G}_\beta}) = \{0\}.$$

Therefore, compositions of these operators have the trivial kernels, i.e.

$$\ker\left((F \otimes I_{E_\beta}) \circ (I_{\mathcal{G}_\beta} \otimes F)\right) = \ker(F \otimes F) = \{0\},$$
$$\ker\left((F^{-1} \otimes I_{\mathcal{G}_\beta}) \circ (I_{E_\beta} \otimes F^{-1})\right) = \ker(F^{-1} \otimes F^{-1}) = \{0\}.$$

Proceeding inductively finite times, we obtain

$$\ker F^{\otimes n} = \ker\big(\underbrace{F \otimes \cdots \otimes F}_{n}\big) = \{0\},$$
$$\ker(F^{-1})^{\otimes n} = \ker\big(\underbrace{F^{-1} \otimes \cdots \otimes F^{-1}}_{n}\big) = \{0\},$$

for all natural n. Note, that the mappings $F^{\otimes n}$, $(F^{-1})^{\otimes n}$ are continuous as tensor products of continuous operators. Since $(F^{\otimes n})^{-1} = (F^{-1})^{\otimes n}$, the mapping $F^{\otimes n} : \mathcal{G}_\beta^{\hat{\otimes} n} \longrightarrow E_\beta^{\hat{\otimes} n}$ is topological isomorphism. Finally, the map $F^{\otimes} : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta)$ is topological isomorphism via the properties of direct sum topology. □

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Ключові слова і фрази: ультрадиференційовна функція, ультрарозподіл, поліноміальна основна функція, теорема типу Пелі-Вінера.

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